EQUATION OF MOTION IN AN INTERNAL CENTRALLY SYMMETRIC FIELD IN GENERAL RELATIVITY

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K. P. STANYUKOVICH

EQUATION OF MOTION IN AN INTERNAL CENTRALLY SYMMETRIC FIELD IN GENERAL RELATIVITY

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Let us consider a convenient form of the equations of motion in an internal centrally symmetric field in general relativity. As we shall see, the equations take their simplest form if we take r and the second coordinate m, which plays the role of a Lagrangian coordinate, as the independent variables.

Let us take the metric of the field in the usual form

$$-ds^2 = -c^2 dt^2 e^{\nu} + dr^2 e^{\lambda} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

Then the basic equations obtained from the conservation laws,

$$T_{i;k}^{k} = [(p+\varepsilon)u_i u^k + \delta_i^k p]_{;k} = [(p+\varepsilon)u_i u^k]_{;k} + p_{,i} = 0$$

will have the form (^{1})

$$\frac{1}{c^2\theta^2}[Au_t + uu_r] - \frac{\omega^2}{c^2}\left[(\ln v)_r + \frac{Au}{c^2}(\ln v)_t\right] = \frac{1}{2u}[A\lambda_t + u\lambda_r] + \frac{\theta^2 T\sigma_r}{W}; \quad (1)$$

$$-[A(\ln v)_t + u(\ln v)_r] + \frac{1}{\theta^2}\left(u_r + \frac{Au}{c^2}u_t\right) + \frac{2u}{r} = \frac{u}{2}\left(\lambda_r + \frac{Au}{c^2}\lambda_t\right); \quad (2)$$

$$A\sigma_t + u\sigma_r = 0. \quad (3)$$

Here $A = e^{(\lambda-\nu)/2}$; $u = A\,dr/dt$; $\theta^2 = 1 - u^2/c^2$; $W = (p+\varepsilon)v$, u is the 3-velocity; p is the pressure; v is the specific volume; $\varepsilon = \rho c^2$ is the energy density; W is the heat content; $\omega^2/c^2 = -(\partial \ln W/\partial \ln v)_\sigma$; ω is the sound velocity; σ is the entropy; T is the temperature.

The field equations, obtained from the equations $R_i^k - \frac{1}{2}\delta_i^{kR} = \chi T_i^k$, have the form

$$(re^{-\lambda})_r = 1 - \frac{\chi r^2}{\theta^2}\left(\varepsilon + p\frac{u^2}{c^2}\right); \tag{4}$$

$$Ar(e^{-\lambda})_t = \frac{\chi u r^2}{\theta^2}(\varepsilon + p); \tag{5}$$

$$(1 + rv_r)e^{-\lambda} = 1 + \frac{\chi r^2}{\theta^2}\left(p + \varepsilon\frac{u^2}{c^2}\right); \tag{6}$$

$$\frac{1}{2}e^{-\lambda}\left[\left(v_{rr} + \frac{1}{2}v_r^2 + \frac{1}{2}(v-\lambda)_r - \frac{v_r\lambda_r}{2}\right) - \frac{A}{c^2}(A\lambda_t)_t\right] = \chi p. \tag{7}$$

There must be 5 independent equations in all. They determine p, u, σ, λ, and v; at the same time, the equation of state of the medium $p = p(\sigma, v)$ must be specified, and the thermodynamic equation $d(\varepsilon v) = T\,d\sigma - p\,dv$ and the identity $\partial(p,v)/\partial(T,\sigma) = 1$ must be used.

After certain transformations, we write 2 independent equations of the system (4)-(7) in the form

$$A\lambda_t + u\left(\lambda_r + (e^{\lambda} - 1)/r + \chi p r e^{\lambda}\right) = 0, \tag{8}$$

$$A(1 + u^2/c^2)\lambda_t + u(\lambda + v)_r = 0. \tag{9}$$

Considering the system of equations (1), (2), (3), (8), we see that these equations do not contain v. Consequently, it is necessary to investigate a system of only 4 equations.

Let us pass from the independent variables $(t;\ r)$ to the variables $(m;\ r)$, where

$$\varkappa\left(\frac{\partial m}{\partial r}\right)_t = \varkappa\,\frac{r^2}{\theta^2}\left(\varepsilon + p\frac{u^2}{c^2}\right) = 1 - (re^{-\lambda})_r, \tag{10}$$

whence

$$m = \int_0^r \frac{r^2}{\theta^2}\left(\varepsilon + p\frac{u^2}{c^2}\right)dr = \frac{r}{\varkappa}(1 - e^{-\lambda}). \tag{11}$$

Equation (18) in the variables (m, r) takes the form

$$\left(A - u\frac{\partial t}{\partial m}\right)\frac{\partial \lambda}{\partial m} + u\frac{\partial t}{\partial m}\left(\frac{\partial \lambda}{\partial r} + \frac{e^{\lambda} - 1}{r} + \varkappa pre^{\lambda}\right) = 0. \tag{12}$$

Equation (10) takes the form

$$\frac{\partial t}{\partial m}\frac{r^2}{\theta^2}\left(\varepsilon + p\frac{u^2}{c^2}\right) + \frac{\partial t}{\partial m} = 0. \tag{13}$$

Equation (11), which we write in the form $e^{-\lambda} = 1 - \varkappa m/r$, determines $\partial e^{-\lambda}/\partial m = -\varkappa/r$, $\partial e^{-\lambda}/\partial r = \varkappa m/r^2$, or

$$\frac{\partial \lambda}{\partial m} = \frac{\varkappa}{r}e^{\lambda}, \qquad \frac{\partial \lambda}{\partial r} = -\frac{\varkappa m}{r^2}e^{\lambda} = -\frac{m}{r}\frac{\partial \lambda}{\partial m}, \tag{14}$$

therefore (12) passes into the equation

$$A - u\frac{\partial t}{\partial r} + u\frac{\partial t}{\partial m}pr^2 = 0. \tag{15}$$

Hence, from (13) we shall have

$$u\,\partial t/\partial m = -A\theta^2/r^2(p+\varepsilon), \qquad u\,\partial t/\partial r = A(\varepsilon + pu^2/c^2)/(p+\varepsilon). \tag{16}$$

Equations (1), (2), (3) take, in the coordinates (m, r), the form

$$\frac{u}{c^2\theta^2}(u_r - pr^2u_m) - \frac{\omega^2}{c^4}\left[(\ln v)_r + \varepsilon r^2(\ln v)_m\right] =$$

$$= \frac{1}{2}(\lambda_r - m^2\lambda_m) + \frac{T}{W}\left[\theta^2\sigma_r + r^2\left(\varepsilon + p\frac{u^2}{c^2}\right)\sigma_m\right],$$

$$-u\left[(\ln v)_r - pr^2(\ln v)_m\right] + \frac{1}{\theta^2}\left[u_r + \varepsilon r^2 u_m\right] + \frac{2u}{r} = \frac{u}{2}(\lambda_r + \varepsilon r^2\lambda_m),$$

$$\sigma_r - pr^2\sigma_m = 0.$$

Transforming the equations, using (14), we shall have

$$\frac{1}{2c^2\theta^2}\left[u_r^2 - pr^2u_m^2\right] - \frac{\omega^2}{c^2}\left[(\ln v)_r + \varepsilon r^2(\ln v)_m\right] +$$

$$+\frac{\varkappa m/r + pr^2}{2(r - \varkappa m)} = \frac{r^2T\sigma_m}{v} = \frac{T\sigma_r}{pv}; \tag{17}$$

$$-\left[(\ln v)_r - pr^2(\ln v)_m\right] + \frac{1}{2\theta^2}\left[(\ln u^2)_r + \varepsilon r^2(\ln u^2)_m\right] +$$

$$+\frac{2}{r} + \frac{\varkappa m/r - \varepsilon r^2}{2(r - \varkappa m)} = 0. \tag{18}$$

As a result we have arrived at a system of only 3 quasilinear equations of the first order.

For isentropic processes, when $\sigma = \text{const}$, the problem reduces to a system of 2 equations.

From equation (16), eliminating A, we shall have

$$\frac{r^2}{\theta^2}\left(\varepsilon + p\frac{u^2}{c^2}\right)\frac{\partial t}{\partial m} + \frac{\partial t}{\partial r} = 0; \tag{19}$$

knowing $\varepsilon = \varepsilon(r,m)$, $u = u(r,m)$ (for a given equation of state), one can (formally) find $t = t(r,m)$, which as a result (again formally) makes it possible to determine $u = u(t,r)$, $\varepsilon = \varepsilon(t,r)$, and, finally, from any

from equation (16) we find

$$A = e^{(\lambda-\nu)/2} = -\frac{r^2}{\theta^2}(\varepsilon + p)u\,\frac{\partial t}{\partial m}; \quad \text{since} \quad e^{\lambda} = \frac{1}{1 - \chi m/r}, \quad \text{then}$$

$$e^{\nu/2} = \frac{1}{\sqrt{1 - \dfrac{\chi m}{r}}}\,\frac{A^2}{r^2 u\,\dfrac{\partial t}{\partial m}(p + \varepsilon)}, \tag{20}$$

which completely solves the self-consistent problem of finding $u, \varepsilon, \lambda, \nu$ for centrally symmetric motions.

Let us now write the characteristic equations of the system of equations (17), (18). Along the lines

$$m' + pr^2 = 0, \qquad \sigma' = 0; \tag{21}$$

here, for example, $\sigma' = d\sigma/dr$, $u' = du/dr$, $m' = dm/dr$, $(\ln\nu)' = d\ln\nu/dr$, etc. Further, expanding the corresponding determinants, we find that along the lines

$$(\varepsilon r^2 - m') = \pm\frac{u}{\omega}(pr^2 + m') \tag{22}$$

the relations

$$(pr^2 + m')\left[\left(\frac{\omega}{c} \pm \frac{u}{c}\right)\left(\frac{\omega}{c}(\ln\nu)' \mp \frac{u'}{c\theta^2}\right)\right] \mp$$

$$\mp \frac{2u\omega}{c^2 r} - \frac{\chi}{2(r-\chi m)}\left[\frac{m}{r}\left(1 \pm \frac{u\omega}{c^2}\right) + r^2\left(p \mp \frac{u\omega}{c^2}\varepsilon\right)\right] = \frac{\sigma' T r^2}{\nu}. \tag{23}$$

In the case $p = 0$, the basic equations (17) and (18) are greatly simplified and are integrated directly (Tolman’ s problem (2)).

The system of equations (1), (2), (3), and (8) also has a solution for the ultra-relativistic approximation, when

$$u/c = 1 - 2\Delta, \qquad \Delta \ll 1.$$

In this case, if terms of order Δ^2 are neglected, the equations take the form

$$\frac{1}{2\Delta}\left[A\Delta_\tau + \Delta_r\right] + \frac{\omega_0^2}{c^2}\left[A(\ln\nu)_\tau + (\ln\nu)_r + \frac{1}{2}[A\lambda_\tau + \lambda_r]\right] = 0,$$

$$[A(\ln\nu)_\tau + (\ln\nu)_r] + \frac{1}{2\Delta}(A\Delta_\tau + \Delta_r) - \frac{2}{r} + \frac{1}{2}[A\lambda_\tau + \lambda_r] = 0, \tag{24}$$

$$A\sigma_\tau + \sigma_r = 0, \qquad A\lambda_\tau + \lambda_r + \frac{e^\lambda - 1}{r} + \chi p r e^\lambda = 0,$$

where $\tau = ct,\ \omega_0^2/c^2 = k - 1$.

It is more convenient to write the first two equations of this system in the form

$$A(\ln\nu)_\tau + (\ln\nu)_r = \frac{2}{(2-k)r},$$

$$\frac{1}{2\Delta}[A\Delta_\tau + \Delta_r] + \frac{2(k-1)}{(2-k)r} = \frac{1}{2}\left[\frac{e^\lambda - 1}{r} + \chi p r e^\lambda\right]. \tag{25}$$

Let us pass to the independent variables (ν, r); then the last two equations of system (24) and the second equation (25), after eliminating the quantity

$$A - \frac{\partial t}{\partial r} = \frac{2\nu}{(2-k)r}\,\frac{\partial t}{\partial \nu} \tag{26}$$

(the first equation (25)), take the form

$$\frac{2\nu}{2-k}\frac{\partial\sigma}{\partial\nu}+r\frac{\partial\sigma}{\partial r}=0, \qquad \frac{2\nu}{2-k}\frac{\partial\lambda}{\partial\nu}+r\frac{\partial\lambda}{\partial r}+e^{\lambda}-1+\chi p r^2 e^{\lambda}=0,$$

$$\frac{1}{2\Delta}\left[\frac{2\nu}{2-k}\frac{\partial\Delta}{\partial\nu}+r\frac{\partial\Delta}{\partial r}\right]+\frac{2(k-1)}{2-k}=\frac{1}{2}\left[e^{\lambda}-1+\chi p r^2 e^{\lambda}\right]. \tag{27}$$

We now transform equation (5) to the coordinates (ν, r); as a result we shall have

$$\frac{\partial\tau}{\partial\nu}\left[r\frac{\partial\lambda}{\partial r}+e^{\lambda}-1-\frac{\chi r^2}{4\Delta}(\varepsilon+p)e^{\lambda}\right]-r\frac{\partial\tau}{\partial r}\frac{\partial\lambda}{\partial\nu}=0. \tag{28}$$

The solution of the resulting system of equations is carried out as follows. From the first equation of system (27) we obtain

$$\sigma=F_1(r,\nu^{-(2-k)/2})=\mathrm{const}\cdot p\nu^{k}. \tag{29}$$

Next we solve the second equation (29) and determine

$$(1-e^{-\lambda})=\frac{1}{r}F_2(r\nu^{-(2-k)/2})+\frac{2-k}{5k-6}\varkappa r^2 p. \tag{30}$$

Further, from the last equation (27) we find

$$\Delta=e^{-\lambda}F_3(r\nu^{-(2-k)/2})r^{-4(k-1)/(2-k)}. \tag{31}$$

From equation (28) we find $\tau=\tau(r,\nu)$, and, finally, equation (26) determines $v=v(r,\nu)$, since

$$A=e^{(\lambda-\nu)/2}=\frac{\partial\tau}{\partial r}+\frac{\partial\tau}{\partial\nu}\frac{2v}{(2-k)r}, \quad \text{whence} \quad e^{-\nu/2}=e^{-\lambda/2}\frac{1}{r}\left[r\frac{\partial\tau}{\partial r}+\frac{\partial\tau}{\partial\nu}\frac{2v}{(2-k)}\right].$$

Detailed calculations here are meaningful only when solving particular problems with already specified functions F_1, F_2, F_3. As a result we obtain exact asymptotic solutions depending on 5 arbitrary functions.

In conclusion, let us make limiting transitions. For $\varkappa=0$ we shall have the equations

$$\frac{1}{2c^2\theta^2}[u_r^2-pr^2u_m^2]-\frac{\omega^2}{c^2}\{(\ln\nu)_r+\varepsilon r^2(\ln\nu)_m\}=\frac{r^2T\sigma_m}{\nu}=\frac{T\sigma_r}{p\nu}; \tag{32}$$

$$-[(\ln\nu)_r - pr^2(\ln\nu)_m] + \frac{1}{2\theta^2u^2}[u_r^2 + \varepsilon r^2 u_m^2] + \frac{2}{r} = 0, \tag{33}$$

which corresponds to special relativity. For $u/c \ll 1$ we obtain the limiting transition to Newton' s theory. Indeed, since in this case $\theta = 1$, $\nu\rho = 1$, $u\,\partial t/\partial r = 1$, $u\,\partial t/\partial M_0 = -1/4\pi\rho r^2$, we shall have[3]

$$uu_r + \omega^2[(\ln\rho)_r + 4\pi\rho r^2(\ln\rho)_{M_0}] + GM_0/r = 4\pi r^2\rho T\sigma_{M_0}, \tag{34}$$

$$u(\ln\rho)_r + u_r + 4\pi\rho r^2 u_{M_0} + 2u/r = 0, \quad \sigma_r = 0. \tag{35}$$

The equations introduced by us are the simplest for analyzing spherically symmetric motions in the general theory of relativity.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.