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ON A CLASS OF GAMES ON THE UNIT SQUARE

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Abstract

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MATHEMATICS

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ON A CLASS OF GAMES ON THE UNIT SQUARE

(Presented by Academician Yu. V. Linnik on 16 X 1967)

In the theory of games on the unit square one often has to solve games whose kernels are composed of pieces of comparatively simple functions. For example, piecewise-linear kernels are composed of pieces of linear functions. In the present article it is clarified what information about the players' strategies can be extracted if it is known that the payoff function has a simple form in some regions of the square, and is arbitrary on the complement of these regions. The theorem proved below makes it possible to reduce certain games with a continuous kernel to matrix games. For its formulation and proof we need some notation.

Let X denote the interval $[0,1]$, and let Y denote a second copy of this interval. We denote the boundary of the interval $X(Y)$ by $\dot{X}(\dot{Y})$. The natural projection of the direct product $X \times Y$ onto $X(Y)$ will be denoted by p_1 (p_2). For any closed set $A \subset X \times Y$, define a family of multivalued mappings Ω_1^A of the interval X into Y, and a family of multivalued mappings Ω_2^A of Y into X, as follows: $T \in \Omega_1^A$ (Ω_2^A) if and only if, for any $x \in X$ $(y \in Y)$ and for any interval $[\alpha, \beta] \subset Y \setminus T(x)$ $(X \setminus T(y))$, the set $p_2^{-1}([\alpha,\beta]) \cap p_1^{-1}(x)$ $(p_1^{-1}([\alpha,\beta]) \cap p_2^{-1}(y))$ is contained in the closure of some one connected component of the complement of the set A. For any set A, the sets Ω_1^A and Ω_2^A are nonempty. For example, if

$$T_1(x) = p_2(p_1^{-1}(x) \cap (A \cup X \times \dot{Y})),$$

$$T_2(y) = p_1(p_2^{-1}(y) \cap (A \cup \dot{X} \times Y)),$$

then it is clear that $T_1 \in \Omega_1^A$ and $T_2 \in \Omega_2^A$.

Proposition 1. If $y \in \Phi_1 \subset Y$, $T_2 \in \Omega_2^A$, $T_2(\Phi_2) \subset \Phi_1 \subset X$, and $[\alpha,\beta] \subset X \setminus \Phi_1$, then $p_1^{-1}([\alpha,\beta]) \cap p_2^{-1}(y)$ is contained in the closure of some connected component of the complement of the set A.

Proof. Since $[\alpha,\beta] \subset X \setminus \Phi_1$, and $T_2(y) \subset \Phi_1$, it must be that $[\alpha,\beta] \subset X \setminus T_2(y)$. Now the assertion of Proposition 1 follows from the definition of Ω_2^A.

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The following proposition is proved analogously.

Proposition 2. If $x \in \Phi_1 \subset X$, $T_1 \in \Omega_1^A$, $T_1(\Phi_1) \subset \Phi_2 \subset Y$, and $[\alpha, \beta] \subset Y \setminus \Phi_2$, then $p_2^{-1}([\alpha, \beta]) \cap p_1^{-1}(x)$ is contained in the closure of some connected component of the complement of the set A.

For each closed set $A \subset X \times Y$ we specify a family M_A of continuous functions on the square, putting $K \in M_A$ if and only if K admits a representation

$$\begin{aligned} K(x,y) = \psi_1(x,y) \cdot f(x) \cdot g(y) + \psi_2(x,y) \cdot f(x) + \\ + \psi_3(x,y) \cdot g(y) + \psi_4(x,y), \end{aligned} \tag{1}$$

where $f(x)$ and $g(y)$ are continuous monotone functions, while $\psi_i(x,y)$, $i = 1,2,3,4$, are constant on each connected component of the complement of the set A and are bounded in the aggregate.

It is easy to see that M_A is nonempty for every A. For example, any linear function on the square is contained in any M_A.

Let us give another example. Consider some piecewise-linear function $K(x,y)$. Denote by A the set of break points of this function (i.e., points in a neighborhood of which $K(x,y)$ is not linear). We shall call this set a lattice. Then M_A contains all piecewise-linear functions with this given lattice and, in particular, $K(x,y)$.

If the functions belonging to M_A are regarded as kernels of games on the square, then M_A determines a set of games on the unit square. We shall denote this set of games by Γ_A. Thus, with each closed set A on the square there are associated the sets of mappings Ω_1^A and Ω_2^A, and the set of games Γ_A. If $\hat{B}$ is some closed subset of X or Y, then by $\hat{B}$ we shall denote the set of probability measures on it.

Theorem. *If there exist closed sets Φ_1 and Φ_2 such that $\dot{X} \subset \Phi_1 \subset X$, $\dot{Y} \subset \Phi_2 \subset Y$, and such $T_1 \in \Omega_1^A$, $T_2 \in \Omega_2^A$ that $T_1(\Phi_1) \subset \Phi_2$ and $T_2(\Phi_2) \subset \Phi_1$, then in every game $\Gamma \in \Gamma_A$ the first player has an optimal strategy whose spectrum is contained in Φ_1, and the second player has an optimal strategy whose spectrum is contained in Φ_2.*

Proof. There are natural embeddings $\hat{\Phi}_1 \subset \hat{X}$ and $\hat{\Phi}_2 \subset \hat{Y}$. If F and G are mixed strategies of the players in the game $\Gamma \in \Gamma_A$, then we shall denote the payoff of the first player by $K(F,G)$.

We first show that for any $F \in \hat{X}$ and $G_\Phi \in \hat{\Phi}_2$ there exists an $F_\Phi \in \hat{\Phi}_1$ such that

$$K(F_\Phi, G_\Phi) \geq K(F, G_\Phi). \tag{2}$$

Let $A_1, A_2, \dots, A_n, \dots$ be the connected components of the complement of the set A, and let $\overline{A}_1, \overline{A}_2, \dots, \overline{A}_n, \dots$ be their closures (since the number of connected

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components of the complement of a closed set on the square is at most countable, we can enumerate them). Let $\psi_i^{\alpha\beta}$ be the constant value that $\psi_i(x,y)$ assumes on A_k. Let $[\alpha,\beta] \subset X \setminus \Phi_1$. Now, by Proposition 1, since $T_2(\Phi_2) \subset \Phi_1$, for any $y \in \Phi_2$ the set $p_1^{-1}([\alpha,\beta]) \cap p_2^{-1}(y)$ is contained in one of the $\overline{A}_i$. Therefore, if by B_i we denote the set of those $y \in \Phi_2$ for which $p_1^{-1}([\alpha,\beta]) \cap p_2^{-1}(y) \subset \overline{A}_i$, then

$$\bigcup_{i=1}^{\infty} B_i = \Phi_2.$$

Putting

$$B_n \setminus \bigcup_{i=1}^{n-1} B_i = C_n,$$

we obtain

$$\bigcup_{n=1}^{\infty} C_n = \Phi_2.$$

The sets C_n are pairwise disjoint and, evidently, measurable.

Define the functions $\psi_i(y)$, $i = 1,2,3,4$, on Φ_2 by setting $\Phi_i(y) = \psi_i^n$ for $y \in C_n$. Since the C_n cover all of Φ_2 and do not intersect, this definition is correct. From the boundedness of $\psi_i(x,y)$ and the measurability of C_n it follows that the $\psi_i(y)$ are integrable. We prove that on $[\alpha,\beta] \times \Phi_2$ the equality

$$K(x,y) = \psi_1(y) \cdot f(x) \cdot g(y) + \psi_2(y) \cdot f(x) + \psi_3(y) \cdot g(y) + \psi_4(y) \qquad (3)$$

holds. Indeed, if $y \in C_n$, then $p_2^{-1}(y) \cap p_1^{-1}([\alpha,\beta]) \subset \overline{A}_n$. Since $K(x,y)$ is continuous, this means that

$$K(x,y) = \lim_{\substack{(x_m,y_m)\to(x,y) \\ (x_m,y_m)\in A_n}} K(x_m,y_m) \quad \text{for } x \in [\alpha,\beta],\ y \in C_n.$$

Now, using representation (1) and passing to the limit, we prove equality (3). Integrating it with respect to $G_\Phi \in \hat{\Phi}_2$, we obtain that $\varphi(x) = \int K(x,y)\,dG_\Phi$ on the interval $[\alpha,\beta]$ has the form $af(x) + b$, where a and b are some numbers. Since we took an arbitrary segment $[\alpha,\beta] \subset X \setminus \Phi_1$, on any interval contained in $X \setminus \Phi_1$ the function

$\varphi(x)$ must have the form $af(x) + b$ (a and b are, generally speaking, different for each segment). By the hypothesis of the theorem, $X \subset \Phi_1$; therefore the set $X \setminus \Phi_1$ is a union of open intervals, and on each closed segment lying in an arbitrary interval the function $\varphi(x)$ is monotone (it is monotone because

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it has the form $af(x) + b$, while $f(x)$ is monotone by the hypothesis of the theorem). From this and from the continuity of $\varphi(x)$ it follows that $\varphi(x)$ attains its maximum value on Φ_1. Suppose the maximum is attained at the point x_0. Put $F_\Phi = I_{x_0}$; then $F_\Phi \in \hat{\Phi}_1$, and for any $F \in \hat{X}$ inequality (2) is valid. Similarly it is proved that for any $G \in \hat{Y}$ and $F_\Phi \in \hat{\Phi}_1$ there exists a $G_\Phi \in \hat{\Phi}_2$ such that

$$K(F_\Phi, G) \geq K(F_\Phi, G_\Phi). \tag{4}$$

Consider the game on the product $\Phi_1 \times \Phi_2$, whose payoff function is $K(x, y)$; since the function $K(x, y)$ is continuous and the sets Φ_1 and Φ_2 are compact, optimal strategies exist in this game (see, for example, ([2])). Let them be F^*_Φ and G^*_Φ. Then, by virtue of equalities (2) and (4), they are also optimal in the game on the whole square. Indeed,

$$K(F^*_\Phi, G) \geq K(F^*_\Phi, G_\Phi) \geq K(F^*_\Phi, G^*_\Phi) \geq$$
$$\geq K(F_\Phi, G^*_\Phi) \geq K(F, G^*_\Phi),$$

whence

$$K(F^*_\Phi, G) \geq K(F^*_\Phi, G^*_\Phi) \geq K(F, G^*_\Phi).$$

The theorem is proved.

Let us give an example in which this theorem is used. Let $K(u)$ be an arbitrary continuous piecewise-linear function on the segment $[-1, 1]$. Let the breakpoints of this function be $-1, u_1, u_2, \dots, u_n, 1$. Suppose, moreover, that the numbers u_i are rational. Then, in order to find the value and some optimal strategies of the game with payoff function $K(x-y)$, it is enough to solve a certain matrix game. Indeed, denoting by B_i the set of points of the square having the form $(x, x+u_i)$, and by A the set $\bigcup_{i=1}^n B_i$, we shall have $K(x-y) \in M_A$. The mapping $T_1(x) = p_2(p_1^{-1}(x) \cap (A \cup X \times \dot{Y}))$ belongs to Ω_1^A, and $T_2(y) = p_1(p_2^{-1}(y) \cap (A \cup \dot{X} \times Y))$ belongs to Ω_2^A. Let M be the least common multiple of the denominators of the u_i. Put

$$\Phi_1 = \bigcup_{i=0}^{M} \left\{ \frac{i}{M} \right\}, \quad \text{and} \quad \Phi_2 = \bigcup_{i=0}^{M} \left\{ \frac{i}{M} \right\}.$$

Then, obviously, $T_1(\Phi_1) \subset \Phi_2$, and $T_2(\Phi_2) \subset \Phi_1$. Therefore, by the theorem, player $i = 1, 2$ has an optimal strategy whose spectrum is contained in Φ_i. Thus, by solving a matrix game of size $(M+1) \times (M+1)$, we shall find the value of the original game and at least one optimal strategy for each player.

As another example, consider the game on the square with payoff function

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$$K(x,y) = \begin{cases} 1 + \dfrac{1}{\delta}(x-y), & \text{if } |x-y| \le \delta,\ x \le y, \\ 1 + \dfrac{1}{\delta}(y-x), & \text{if } |x-y| \le \delta,\ x \ge y, \\ 0, & \text{if } |x-y| > \delta. \end{cases}$$

Earlier N. N. Vorob' ev indicated to me an optimal strategy for each player, and the spectrum of the optimal strategies was concentrated on the set

$$\Phi = \bigcup_{k=0}^{n} \{k\delta\} \cup \bigcup_{k=0}^{n} \{1 - k\delta\},$$

where $n = [1/\delta]$. If we denote by A

the set of points of the square having the form $(x,x) \cup (x, x+\delta) \cup (x, x-\delta)$, and T_1 and T_2 are defined by the formulas of the preceding example, then, obviously, $T_1(\Phi) \subset \Phi$ and $T_2(\Phi) \subset \Phi$. Thus, it follows from the theorem that one could have sought in advance optimal strategies whose spectrum is Φ.

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References

[1] C. Karlin, *Mathematical Methods in Game Theory, Programming, and Economics*, Moscow, 1964. [2] I. L. Glicksberg, in: *Infinite Antagonistic Games*, Moscow, 1963, p. 497.

Note: Figure translations are in progress. See original paper for figures.

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