THE BASIC STRUCTURE OF A \(B\)-SPACE AND PROPERTIES OF THE SPHERE INVARIANT UNDER ISOMORPHISMS

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 513.88:513.83

MATHEMATICS

V. D. MILMAN

THE BASIC STRUCTURE OF A *B*-SPACE AND PROPERTIES OF THE SPHERE INVARIANT UNDER ISOMORPHISMS

(Presented by Academician L. V. Kantorovich on 27 V 1967)

We continue in the present note the study of the characteristics of the sphere introduced in (^{1}) and establish their connection with properties of subspaces, bases, and reflexivity. Preliminary results on isomorphic norms, used later, are given. In Sec. 3 numerical characteristics of the space are introduced which do not depend on points of the sphere.

We shall adopt the following notation: B is a Banach space; if $r(x)$ is an equivalent norm, then B_r denotes the space B considered in the norm $r(x)$; E^n is a closed subspace of B of defect n, and E_n is an n-dimensional subspace of B; $S(E) = \{x \in E : \|x\| = 1\}$. By $E(M)$ we denote the closed linear span in B of the set M. A basic sequence $\{x_k\}_1^\infty \subset S(B)$ (i.e., a basis in $E(\{x_k\}_1^\infty)$) is called **0-orthogonal** if for every $\varepsilon > 0$ there is an n such that, for any n_1, n_2 and m_1, m_2 such that $n_2 \geq n_1 + m_1 \geq n_1 \geq n$,

$$(1-\varepsilon)\left\|\sum_{n_1+m_1}^{n_2} a_{kx}k\right\| \leq \left\|\sum_{n_1}^{n_2} a_{kx}k\right\| \leq (1+\varepsilon)\left\|\sum_{n_1}^{n_2+m_2} a_{kx}k\right\|.$$

The space B_1 is ε-isometric to B (notation $B_1 \overset{\varepsilon}{\simeq} B$) if there exists an operator T establishing an isomorphism between B_1 and B for which $\|T\| \cdot \|T^{-1}\| \leq 1 + \varepsilon$; $B_1 = B$ denotes an isometry of B_1 and B. We shall say that a space B is **saturated** with subspaces from a family $\mathfrak{M}$ if for every infinite-dimensional $B_1 \subset B$ there are $E \in \mathfrak{M}$ and $E \subset B_1$.

The richness of a certain set $\mathfrak{A} \subset S(B)$ will be described in two ways: (I) $\forall E^n \subset B$, $\mathfrak{A} \cap E^n \neq \emptyset$; (II) for every infinite-dimensional $B_1 \subset B$, $\mathfrak{A} \cap B_1 \neq \emptyset$.

1. Isomorphic norms

Theorem 1. *Let B be isomorphic to l_1 (or c_0). Then for every $\varepsilon > 0$ there exists in B a subspace B_1, ε-isometric to l_1 (respectively ε-isometric to c_0).*

Before formulating the corollary of the theorem, we introduce the following concept. Let $\varphi(x)$ be a real continuous function for $x \in S(B)$. We shall call a number a a **point of the spectrum** $(*)$ of $\varphi(x)$ if for every $\varepsilon > 0$ there exists an infinite-dimensional subspace $B_1 \subset B$ such that $|\varphi(x) - a| < \varepsilon$ for $x \in S(B_1)$. We shall call the number a a **point of the spectrum** $(**)$ of $\varphi(x)$ if for every $\varepsilon > 0$ and N there exists a subspace $E_N \subset B$ such that $|\varphi(x) - a| < \varepsilon$ for $x \in S(E_N)$.

By $R[S(B)]$ we denote the closure* in $R[S(B)]$ of continuous superpositions of functions on $S(B)$ from the ring generated by all restrictions—

---

* $R[A]$ denotes the space of uniformly continuous functions on A with norm $\|\varphi\| = \max_{x \in A} |\varphi(x)|$.

functions that are the restriction to $S(B)$ of convex or concave functions in B.

Corollary 1. The spectrum $(*)$ of any function $\varphi(x) \in \bar{R}[S(l_1)]$ is nonempty. The same is true for $\varphi(x) \in \bar{R}[S(c_0)]$.

I do not know how to prove an analogous assertion for an arbitrary B-space, although the following holds.

Theorem 2. The spectrum $(**)$ of any function $\varphi(x) \in R[S(B)]$ is nonempty. (B is an arbitrary Banach space.)

We illustrate the notion of spectrum introduced above by the following example. The **isometry spectrum** $(*)$ (or $(**)$) of a continuous bounded, but generally speaking nonlinear, operator $A : S(B) \to B$ will mean the spectrum $(*)$ (respectively $(**)$) of the function

$$\varphi(x) = \|Ax\|, \qquad x \in S(B).$$

If the operator A is completely continuous, then its isometry spectrum $(*)$ (and $(**)$) consists only of 0. However, the converse is not true. The largest point of the spectrum $(*)$ of the function (Ax, x), where A is a self-adjoint operator and $x \in S(H)$, is the largest point of the continuous spectrum of the operator A.

In what follows we shall operate with the following characteristics of the unit sphere (for details on them see (^{1}); there their values for certain spaces are also indicated):

$$\beta^0(\varepsilon; x, B) = \sup_{n;\ E^n \subset B} \inf_{y \in S(E^n)} \|x/\|x\| + \varepsilon y\| - 1;$$

$$\delta^0(\varepsilon; x, B) = \inf_{n;\ E^n \subset B} \sup_{y \in S(E^n)} \|x/\|x\| + \varepsilon y\| - 1.$$

Theorem 3. Let, for an equivalent norm $r(x)$,

$$c_1 r(x) \leq \|x\| \leq c_2 r(x).$$

Then for every $\chi > 0$ there are sets $\mathfrak{A}_1(\chi) \subset S(B)$ and $\mathfrak{A}_2(\chi) \subset S(B)$ such that, for

$$x \in \mathfrak{A}_1(\chi),$$

$$\beta^0\left(\frac{c_1}{c_2}\varepsilon; x, B_r\right) < \beta^0(\varepsilon, x, B) + \chi; \qquad \delta^0\left(\frac{c_1}{c_2}\varepsilon; x, B_r\right) < \delta^0(\varepsilon, x, B) + \chi,$$

and, for

$$x \in \mathfrak{A}_2(\chi),$$

$$\beta^0(\varepsilon; x, B_r) \geq \beta^0\left(\frac{c_1}{c_2}\varepsilon; x, B\right) - \chi; \qquad \delta^0(\varepsilon; x, B_r) \geq \delta^0\left(\frac{c_1}{c_2}\varepsilon; x, B\right) - \chi.$$

Moreover, the richness of the sets $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is described by property (I) (see the introduction).

Let us note that if

$$\beta^0(\varepsilon; x, B) \equiv \delta^0(\varepsilon; x, B)^*,$$

then the sets $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are rich in the sense of (II).

2. Subspaces of spaces with unconditional bases will be called spaces of type D.

Theorem 4. If B is of type D and $\beta^0(\varepsilon_0; x, B) = 0$ ($\varepsilon_0 > 0$), then there exists $B_1 \subset B$, $B_1 \simeq c_0$.

Corollary 2. (A. Pełczyński (2)) $L[0,1]$ is not a subspace of a space with an unconditional basis.

Theorem 5. If B is of type D and

$$\lim_{\varepsilon \to 0} \delta^0(\varepsilon; x, B)/\varepsilon = 1$$

(uniformly in $x \in S$), then there exists $B_1 \subset B$, $B_1 \simeq l_1$.

3. On certain numerical functions corresponding to Banach spaces. The dependence of the functions $\beta^0(\varepsilon; x, B)$ and $\delta^0(\varepsilon; x, B)$, studied above (and in (1)), on $x \in S(B)$ worsens some results. Moreover, it is clear that, in the circle of questions under consideration, these functions can be considered only in certain subspaces with finite defect. In this connection let us introduce four functions of B (no longer depending on $x \in S(B)$): $\beta^0\beta^0(\varepsilon; B)$, $\delta^0\beta^0(\varepsilon; B)$, $\beta^0\delta^0(\varepsilon; B)$, and $\delta^0\delta^0(\varepsilon; B)$, which are obtained from $\beta^0(\varepsilon; x, B)$ and $\delta^0(\varepsilon; x, B)$ by one of the following two processes:

$$\beta^0 f \equiv \sup_{n,\ E^n \subset B} \inf_{x \in S(E^n)} f(x); \qquad \delta^0 f \equiv \inf_{n,\ E^n \subset B} \sup_{x \in S(E^n)} f(x).$$

---

* For example, for l_p,

$$\beta^0(\varepsilon; x, l_p) \equiv \delta^0(\varepsilon; x, l_p) \equiv (1 + \varepsilon^p)^{1/p} - 1 \sim \varepsilon^p/p.$$

Proposition 1. Let B_1 be an infinite-dimensional subspace of B. The following relations hold:

$$1) \quad \beta^0\beta^0(\varepsilon; B) \le \frac{\beta^0\delta^0(\varepsilon; B)}{\delta^0\beta^0(\varepsilon; B)} \le \delta^0\delta^0(\varepsilon; B);$$

$$2) \quad \beta^0\beta^0(\varepsilon; B) \le \beta^0\beta^0(\varepsilon; B_1) \le \delta^0\delta^0(\varepsilon; B_1) \le \delta^0\delta^0(\varepsilon; B).$$

For the functions $\beta^0\beta^0(\varepsilon; B)$ and $\delta^0\delta^0(\varepsilon; B)$ there are analogues of Theorems 4 and 4a from (^{1}). We shall not formulate these results here, but shall give only a consequence of them.

Theorem 6. Suppose there exist $C_2 \ge C_1 > 0$ and functions $\psi(\varepsilon)$ such that

$$C_1\psi(\varepsilon) \le \beta^0\beta^0(\varepsilon; B) \le \delta^0\delta^0(\varepsilon; B) \le C_2\psi(\varepsilon).$$

Then in B there exists an unconditional basic sequence $\{x_k\}_1^\infty \subset S(B)$, and the convergence of the series

$$\sum_1^\infty a_{kx}k$$

is equivalent to the convergence of the series

$$\sum_1^\infty \psi(|a_k|).$$

Item c) of the following assertion is a trivial consequence of Theorem 6.

Theorem 7. a) If there exists $\varepsilon_0 > 0$ such that $\beta^0\delta^0(\varepsilon_0; B) = 0$, then $\exists B_1 \subset B$, $B_1 \simeq c_0$. If $\delta^0\beta^0(\varepsilon; B) \ge a\varepsilon$ $(a > 0)$, then $\exists B_1 \subset B$, $B_1 \simeq l_1$.

b) If $\exists B_1 \subset B$, $B_1 \simeq c_0$, then $\beta^0\beta^0(\varepsilon; B) = 0$ for $\varepsilon \le 1$. If $\exists B_1 \subset B$, $B_1 \simeq l_1$, then $\delta^0\delta^0(\varepsilon; B) = \varepsilon$ for $\varepsilon \le 1$.

c) If $\delta^0\delta^0(\varepsilon_0; B) = 0$ $(\varepsilon_0 > 0)$, then B is saturated with subspaces isomorphic to c_0. If $\beta^0\beta^0(\varepsilon; B) \ge a\varepsilon$ $(a > 0)$, then B is saturated with subspaces isomorphic to l_1.

The following theorem points out some properties of the infinite-dimensional sphere (items a) and b)) and their preservation under isomorphisms (items c) and d)); here Theorem 3 is used.

Theorem 8. a) If $\delta^0\beta^0(\varepsilon_0; B) = 0$, $\varepsilon_0 > 0$, then $\beta^0\beta^0(\varepsilon; B) = 0$ for $\varepsilon \le 1$.

b) If

$$\lim_{\varepsilon\to 0} \beta^0\delta^0(\varepsilon; B)/\varepsilon = a > 0,$$

then $\delta^0\delta^0(\varepsilon; B) = \varepsilon$ for $\varepsilon \leq 1$.

c) If $\delta^0\beta^0(\varepsilon_0; B) = 0$, $\varepsilon_0 > 0$, then for every B_1 isomorphic to B,

$$\beta^0\beta^0(\varepsilon_0; B_1) = 0.$$

d) If

$$\lim_{\varepsilon\to 0} \beta^0\delta^0(\varepsilon; B)/\varepsilon = a > 0,$$

then for every B_1 isomorphic to B,

$$\lim_{\varepsilon\to 0} \delta^0\delta^0(\varepsilon; B_1)/\varepsilon \geq a.$$

4. Connection with properties of bases. A basis $\{x_k\}_1^\infty \subset B$ is called **boundedly complete** if, for every numerical sequence $\{a_k\}_1^\infty$ such that

$$\sup_n \left\| \sum_1^n a_{kx} k \right\| < M,$$

the series

$$\sum_1^\infty a_{kx} k$$

converges.

Theorem 9. If $\beta^0(\varepsilon_0; x, B) = 0$ $(\varepsilon_0 > 0)$, then B contains a boundedly complete basic sequence and B has no boundedly complete basis.

If

$$\lim_{\varepsilon\to 0} \delta^0(\varepsilon; x, B)/\varepsilon = 1 \quad \text{(uniformly in } x \in S(B)\text{)},$$

then in B there exists a basic sequence $\{x_k\}_1^\infty \subset S(B)$ from which no weakly convergent subsequence can be selected.

Thus, from Theorem 9 and the connection between properties of bases and reflexivity (([4],[5]), see, for example, ([3]), pp. 121, 122) we obtain

Corollary 3. For reflexivity of the space B it is necessary that for every $\varepsilon_0 > 0$ in each E^n there exist x_1 and x_2 such that

$$\beta^0(\varepsilon_0; x_1, B) > 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \delta^0(\varepsilon; x_2, B)/\varepsilon < 1.$$

We note that no sufficient conditions for reflexivity can be given solely in terms of β^0 and δ^0, as the following example shows: for James' s space J (see, for example, (^{3}), p. 123) we have

$$\beta^0(\varepsilon; x, J) \equiv \delta^0(\varepsilon; x, J) = (1+\varepsilon^2)^{1/2} - 1 \sim \varepsilon^2/2.$$

At the same time, the criterion of saturation-

ness with reflexive subspaces is given in Theorem 10 (the above-mentioned results of James are also used here).

Theorem 10. If $\beta^0\beta^0(1; B) > 0$, then B is saturated by subspaces with a boundedly complete basis, and hence (by a theorem of Alaoglu; see (3), p. 120), by ε-isometric conjugates. If there is an $\mathfrak{E}\varepsilon_0 > 0$ such that $\delta^0\delta^0(\varepsilon_0; B) < \varepsilon_0$, then B is saturated by subspaces with shrinking* bases (in particular, by subspaces with separable conjugates). If simultaneously $\beta^0\beta^0(1; B) > 0$ and there is an $\mathfrak{E}\varepsilon_0 > 0$ such that $\delta^0\delta^0(\varepsilon_0; B) < \varepsilon_0$, then B is saturated by reflexive subspaces.

5. β- and δ-characteristics connected with a family of subspaces. Let $\mathfrak{B} = \{B_\alpha\}$ be some collection of subspaces of the space B, satisfying the conditions: 1) if $B_i \in \mathfrak{B}$, $(i = 1, 2)$, then $\mathfrak{E}B_3 \subset B_1 \cap B_2$ and $B_3 \in \mathfrak{B}$; 2) for any $B_1 \in \mathfrak{B}$, $\mathfrak{E}B_2 \subset B_1$, but $B_2 \neq B_1$, and $B_2 \in \mathfrak{B}$. Define the functions $(x \neq 0)$

$$\delta(\varepsilon; x, \mathfrak{B}) = \inf_{E\in\mathfrak{B}} \sup_{y\in S(E)} \left\| \frac{x}{\|x\|} + \varepsilon y \right\| - 1,$$

$$\beta(\varepsilon; x, \mathfrak{B}) = \sup_{E\in\mathfrak{B}} \inf_{y\in S(E)} \left\| \frac{x}{\|x\|} + \varepsilon y \right\| - 1.$$

Obviously, for $\mathfrak{B}$ consisting of all subspaces B of finite defect, we obtain respectively $\delta^0(\varepsilon; x, B)$ and $\beta^0(\varepsilon; x, B)$.

Principal examples of families $\mathfrak{B}$. a) $\mathfrak{B}^1(E)$ is the family of all subspaces B of finite defect in an infinite-dimensional subspace E; b) $\mathfrak{B}^2(E)$ is the family of all subspaces of finite defect in B and containing E; c) $\mathfrak{B}^3(\{x_k\}_1^\infty) = \{E(\{x_k\}_n^\infty)\}_{n=1}^\infty$; d) $\mathfrak{B}^4(\{x_k\}_1^\infty : E^n \in \mathfrak{B}^4$, if $\mathfrak{E}N$ is such that $E(\{x_k\}_N^\infty) \subset E^n$.

Remark 1. For some families $\mathfrak{B}$, $\beta(\varepsilon; x, \mathfrak{B})$ may also take negative values. However, in the examples indicated above, if only $\{x_k\}_1^\infty$ is a 0-orthogonal basic sequence, $\beta \geq 0$.

Remark 2. In examples b) and d), and also c) when $E(\{x_k\}_1^\infty) = B$,

$$\beta(\varepsilon; x, \mathfrak{B}) \leq \beta^0(\varepsilon; x, B) \leq \delta^0(\varepsilon; x, B) \leq \delta(\varepsilon; x, \mathfrak{B}).$$

Remark 3. For the functions δ and β introduced here, Theorems 1 and 4 (as well as the corollaries from them) from (1) hold, and also all assertions of §§ 3 and 4 of the present article, with the exception of Theorem 10, in which the word "saturated" must everywhere be replaced by "there exists."

When considering the class of isomorphic spaces and clarifying the connection of the functions β and δ with properties of bases and reflexivity, the passage to the families of examples c) and d) strengthens the preceding results.

Theorem 11. a) If B is nonreflexive, then there exists a basic sequence $\{x_k\}^\infty \subset S(B)$ such that $\beta(1; x_k, \mathfrak{B}^3(\{x_k\}^\infty)) \to 0$ $(k \to \infty)$ and $\delta(\varepsilon; x_k, \mathfrak{B}^3(\{x_k\})) \geq (1 - \varkappa_k)\varepsilon$ $(\varepsilon \leq 1)$, where $\varkappa_k \to 0$ $(k \to \infty)$.

b) If, for some $\mathfrak{B}$, there is an $\mathfrak{E}\varepsilon_0 > 0$ such that $\beta(\varepsilon_0; x, \mathfrak{B}) = 0$ or

$$\lim_{\varepsilon \to 0} \delta^0(\varepsilon; x, \mathfrak{B})/\varepsilon = 1,$$

then the space B is nonreflexive.

Part b) of the theorem is a reformulation, for the case of the family $\mathfrak{B}$, of Corollary 3 and is given separately for comparison with part a).

Institute of Chemical Physics
Academy of Sciences of the USSR

Received
13 V 1967

CITED LITERATURE

1. V. D. Milman, DAN, 177, No. 3, 514 (1967).
2. A. Pełczyński, Colloquium Math., 8, No. 2, 199 (1961).
3. M. M. Day, *Normed Linear Spaces*, Moscow, 1961.
4. R. C. James, Ann. Math. (2), 52, 518 (1950).
5. R. C. James, Proc. Nat. Acad. Sci., U.S.A., 37, 174 (1951).

* For the definition and properties of shrinking bases see (3), p. 119.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.