Reports of the Academy of Sciences of the USSR

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1968. Volume 183, No. 5

UDC 517.5

MATHEMATICS

P. E. SOBOLEVSKII

ON QUASILINEAR EQUATIONS IN A BANACH SPACE

(Presented by Academician I. N. Vekua on 1 III 1968)

1. In a Banach space E we consider the problem

$$v'(t) = A(t)v(t) + f[t, v(t)] \quad (0 \leq t \leq T), \qquad v(0) = v_0. \tag{1}$$

A solution of this problem is a function $v(t)$ satisfying (1) and such that the functions $v'(t)$ and $A(t)v(t)$ are continuous on $[0, T]$. It is assumed that $A(t)$, for each t in $[0, T]$, is the generating operator of a strongly continuous contraction semigroup. The domain D of the operator $A(t)$ does not depend on t, and the operator-function $A(t)A^{-1}(0)$ is strongly continuously differentiable. If the operator A does not depend on t, then it is assumed only that it generates a strongly continuous semigroup.

In work (^{1}) the unique local solvability of problem (1) was proved under the assumption that the derivatives f'_t and f'_v exist for all t in $[0, T]$ and v in E and satisfy a Lipschitz condition with respect to v. In (^{2}) this fact is established under the assumption of mere continuity of the derivatives f'_t and f'_v. A strengthening of the result from (^{1}) in another direction was obtained in (^{3}), where it is assumed that the operator f acts (with respect to v) not from E into E, but only from D into E. This makes it possible to consider also unbounded nonlinear operators f.

In work (^{1}) generalized solutions of problem (1) were also studied, and the existence of at least one generalized solution was proved under the assumption that the operator f is the sum of a Lipschitz operator and a completely continuous operator. Below, in § 2, a theorem is given which generalizes in this respect the result of (^{1}) for solutions of problem (1), and not only for its generalized solutions. In § 3 an analogous theorem is given for second-order differential equations.

2. **Theorem 1.** *Let the operator $f(t, v)$ act from $[0, T] \times D$ into E. Here D is regarded as a Banach space with norm $\|v\|_D = \|A(0)v\|_E$. Let the operator $f(t, v)$ have derivatives $f'_t(t, v)$ and $f'_v(t, v)$. Let $f'_t(t, v) = f_{1t}(t, v) + f_{2t}(t, v)$. Each of the operators $f_{it}(t, v)$ is continuous jointly in the variables. The operator $f_{1t}(t, v)$ satisfies the Lipschitz condition*

$$\|f_{1t}(t, v_1) - f_{1t}(t, v_2)\|_E \leq C(R)\|v_1 - v_2\|_D, \qquad \|v_i\|_D \leq R.$$

For any fixed t, the operator $f_{2t}(t, v)$ maps every set bounded in D into a set compact in E.

Let the Fréchet derivative $f'_v(t, v)$, which for each fixed t and v is a linear continuous operator acting from D into E, admit a continuous extension to a linear continuous operator acting in E. We shall retain the same notation $f'_v(t, v)$ for this extension. Let $f'_v(t, v) = f_{1v}(t, v) + f_{2v}(t, v)$. For any fixed h in E, the operators $f_{iv}(t, v)$ (from $[0, T] \times D$ into E) are continuous jointly in the variables t and v. The operator $f_{1v}(t, v)h$ satisfies the Lipschitz condition

$$\|[f_{1v}(t, v_1) - f_{1v}(t, v_2)]h\|_E \leq C(R)\|h\|_E\|v_1 - v_2\|_D, \qquad \|v_i\|_D \leq R.$$

For any fixed t, the operator $f_{2v}(t, v)h$ maps every set bounded in D into a set compact in E.

Let v_0 belong to D. Then problem (1) has a unique solution, defined on some segment $[0, t_0] \subset [0, T]$.

In the case where $f_{2t} \equiv 0$ and $f_{2v} \equiv 0$, Theorem 1 was proved in (3). Here, in order to shorten the exposition, its proof is given in the case where $A(t) \equiv A$, $f(t, v) \equiv f(v)$, $f'(v) = f_{2v}(v)$. Thus, let $v(t)$ be a solution of the problem

$$v'(t) = Av(t) + f[v(t)], \qquad v(0) = v_0. \tag{2}$$

Then

$$v(t) = \exp\{tA\}v_0 + \int_0^t \exp\{(t - s)A\}f[v(s)]\,ds. \tag{3}$$

From (3) it follows that

$$v'(t) = \exp\{tA\}Av_0 + \exp\{tA\}f(v_0) +$$

$$+ \int_0^t \exp\{(t - s)A\}f'[v(s)]v'(s)\,ds. \tag{4}$$

Consider the integral equation

$$z(t) = \exp\{tA\}Av_0 + \exp\{tA\}f(v_0) +$$

$$+ \int_0^t \exp\{(t-s)A\}f'[v(s)]z(s)\,ds. \tag{5}$$

Its continuous solution is unique and has the form

$$z(t) = \exp\{tA\}Av_0 + \exp\{tA\}f(v_0) +$$

$$+ \int_0^t \exp\{(t-s)A\}\varphi[s, v(s)]\,ds, \tag{6}$$

where

$$\varphi[t, v(t)] = f'[v(t)]z[t, v(t)],$$

$$z[t, v(t)] = \sum_{n=0}^{+\infty} z_n[t, v(t)],$$

$$z_0[t, v(t)] = \exp\{tA\}Av_0 + \exp\{tA\}f(v_0),$$

$$z_{n+1}[t, v(t)] = \int_0^t \exp\{(t-s)A\}z_n[s, v(s)]\,ds. \tag{7}$$

From (4) it then follows that

$$v'(t) = \exp\{tA\}Av_0 + \exp\{tA\}f(v_0) + \int_0^t \exp\{(t-s)A\}\varphi[s, v(s)]\,ds. \tag{8}$$

Finally, integrating (8), we obtain

$$v(t) = \exp\{tA\}v_0 + [\exp\{tA\} - I]A^{-1}f(v_0) +$$

$$+ \int_0^t [\exp\{(t-s)A\} - I]A^{-1}\varphi[s, v(s)]\,ds. \tag{9}$$

Making in equation (9) the substitution $w(t) = Av(t)$, we reduce the problem to finding a continuous solution in E of the equation

$$w(t) = \exp\{tA\}Av_0 + [\exp\{tA\} - I]f(v_0) +$$

$$+ \int_0^t [\exp\{(t-s)A\} - I]\varphi[s, A^{-1}w(s)]\,ds. \tag{10}$$

We shall regard (10) as an operator equation in the space $C([0, t_0], E)$ of continuous functions $w(t)$ defined on $[0, t_0]$ with values ···

chains in E. Write it in the form $w = Kw$. For sufficiently small t_0 the operator K maps some ball of the space C into itself and is continuous in C. From Mazur' s theorem it follows that the operator K transforms any bounded set of functions $N = \{w(t)\}$ in C into a set of such functions $KN = \{Kw(t)\}$ that the set of their values for each fixed t is compact in E.

Finally, let us prove that KN is a set of equicontinuous functions. For this, with the aid of the Schauder projection operator (see, for example, (^{4})) P_n, we construct a sequence of operators K_n converging uniformly on N to K. Finally, P_n (and correspondingly K_n) is slightly deformed so that the basis vectors belong to D. Then $\widetilde{K}_n N$, for the resulting $\widetilde{K}_n$, will be a set of equicontinuous functions. Thus the Schauder principle is applicable to the equation $w = Kw$. Consequently, problem (2) has at least one solution $v(t)$, defined on some segment $[0, t_0]$.

The smoothness of the operator f makes it possible to prove uniqueness, although it does not allow the problem to be reduced to the contraction mapping principle. If the operator $f'(v)$ were the sum of a Lipschitz operator $f_1(v)$ and a completely continuous operator $f_2(v)$, then the equation with the completely continuous operator could be reached in a somewhat different way. From equality (4) we pass to the equality

$$\begin{aligned} v(t) = {} & \exp\{tA\}v_0 + [\exp\{tA\} - I]A^{-1}f(v_0) + \\ & + \int_0^t [\exp\{(t-s)A\} - I]A^{-1}f'[v(s)]v'(s)\,ds, \end{aligned} \tag{11}$$

and then consider the system, following from (4) and (11),

$$\begin{aligned} z(t) = {} & \exp\{tA\}Av_0 + \exp\{tA\}f(v_0) + \\ & + \int_0^t \exp\{(t-s)A\}(f_1[A^{-1}w(s)] + f_2[v(s)])z(s)\,ds, \\ w(t) = {} & \exp\{tA\}Av_0 + [\exp\{tA\} - I]f(v_0) + \\ & + \int_0^t [\exp\{(t-s)A\} - I](f_1[A^{-1}w(s)] + f_2[v(s)])z(s)\,ds, \end{aligned} \tag{12}$$

where $z(t) = v'(t)$, $w(t) = Av(t)$. System (12) can be solved uniquely, since $z(t)$ enters linearly, while $w(t)$ enters under the sign of the Lipschitz operator f_1.

3. For the investigation of quasilinear equations of higher orders one may pass to a system of first-order equations, and then apply results ($^{1-3}$) or Theorem 1. Thus one may proceed in the investigation of the problem

$$\begin{gathered} v''(t) + A(t)v(t) = f[t, v(t), v'(t)] \qquad (0 \le t \le T), \\ v(0) = v_0, \quad v'(0) = v'_0 \end{gathered} \tag{13}$$

in a real Hilbert space H with a positive definite self-adjoint operator $A(t)$. Its solution is a function $v(t)$ satisfying (13) and such that the functions $v''(t)$ and $A(t)v(t)$ are continuous on $[0, T]$. This problem, as S. G. Krein showed, can be reduced to a system of first-order equations with a matrix operator satisfying the conditions of item 1, but for this one has to consider the problem in a complex Hilbert space.

If in problem (13) the operator $A(t) = A(0) = A$, then problem (13) is equivalent to the problem

$$v(t) = \cos A^{1/2}t\,v_0 + A^{-1/2}\sin A^{1/2}t\,v'_0 + \int_0^t A^{-1/2}\sin A^{1/2}(t-s)f[s, v(s), v'(s)]\,ds \tag{14}$$

in the same real Hilbert space H. The methods developed in (5) make it possible to reduce also the problem with a variable operator $A(t)$ to an analogous equation.

Theorem 2. Let the domain of definition D of the operator $A(t)$ not depend on t, and let the operator-function $A(t)A^{-1}(0)$ be strongly continuously differentiable. Let the operator $f(t, v, w)$ act from $[0, T] \times D \times D_{1/2}$ into H. Here $D_{1/2}$ denotes the domain of definition of the operator $A^{1/2}(0)$ with norm $\|w\|_{D_{1/2}} = \|A^{1/2}(0)w\|_H$. Let the operator $f(t, v, w)$ have the derivatives $f'_t(t, v, w)$, $f'_v(t, v, w)$, $f'_w(t, v, w)$. Let $f'_t(t, v, w) = f_{1t}(t, v, w) + f_{2t}(t, v, w)$. Each of the operators $f_{it}(t, v, w)$ is continuous in the aggregate of variables. The operator $f_{1t}(t, v, w)$ satisfies the Lipschitz condition

$$\|f_{1t}(t,v_1,w_1)-f_{1t}(t,v_2,w_2)\|_H \le C(R)(\|v_1-v_2\|_D+\|w_1-w_2\|_{D_{1/2}}),$$

$$\|v_i\|_D \le R, \qquad \|w_i\|_{D_{1/2}} \le R.$$

The operator $f_{2t}(t,v,w)$, for any fixed t, maps every set bounded in $D \times D_{1/2}$ into a set compact in H. Let the Fréchet derivative $f'_v(t,v,w)$, which for each fixed t,v,w is a linear continuous operator acting from D into H, admit a continuous extension to a linear continuous operator acting from $D_{1/2}$ into H. We retain the same notation $f'_v(t,v,w)$ for this extension. Let $f'_v(t,v,w) = f_{1v}(t,v,w)+f_{2v}(t,v,w)$. For any fixed $h \in D_{1/2}$, the operators $f_{iv}(t,v,w)h$ (from $[0,T]\times D\times D_{1/2}$ into H) are continuous in the aggregate of variables t,v,w. The operator $f_{1v}(t,v,w)h$ satisfies the Lipschitz condition

$$\|[f_{1v}(t,v_1,w_1)-f_{1v}(t,v_2,w_2)]h\|_H \le$$

$$\le C(R)\|h\|_{D_{1/2}}(\|v_1-v_2\|_D+\|w_1-w_2\|_{D_{1/2}}), \qquad \|v_i\|_D \le R, \quad \|w_i\|_{D_{1/2}} \le R.$$

The operator $f_{2v}(t,v,w)h$, for any fixed t, maps every set bounded in $D\times D_{1/2}$ into a set compact in H.

Let the Fréchet derivative $f'_w(t,v,w)$, which for each fixed t,v,w is a linear continuous operator acting from $D_{1/2}$ into H, admit a continuous extension to a linear continuous operator acting in H. We retain the same notation $f'_w(t,v,w)$ for this extension. Let $f'_w(t,v,w) = f_{1w}(t,v,w)+f_{2w}(t,v,w)$. For any fixed h from H, the operators $f_{iw}(t,v,w)$ (from $[0,T]\times D\times D_{1/2}$ into H) are continuous in the aggregate of variables t,v,w. The operator $f_{1w}(t,v,w)$ satisfies the Lipschitz condition

$$\|[f_{1w}(t,v_1,w_1-f_{1w}(t)v_2,w_2)]h\|_H \le$$

$$\le C(R)\|h\|_H(\|v_1-v_2\|_D+\|w_1-w_2\|_{D_{1/2}}), \qquad \|v_i\|_D \le R, \quad \|w_i\|_{D_{1/2}} \le R.$$

The operator $f_{2w}(t,v,w)h$, for any fixed t, maps every set bounded in $D\times D_{1/2}$ into a set compact in H. Let $v_0 \in D$, $v'_0 \in D_{1/2}$.

Then problem (13) has a unique solution, defined on some segment $[0,t_0] \subset [0,T]$.

Voronezh State University

Received
23 II 1968

CITED LITERATURE

1. M. A. Krasnosel' skii, S. G. Krein, P. E. Sobolevskii, DAN, 111, No. 1, 19 (1956).
2. V. G. Raskin, P. E. Sobolevskii, Siberian Math. Journal, 8, No. 1, 70 (1967).
3. J. E. Segal, Ann. of Math., 78, 333 (1963).
4. L. A. Lyusternik, V. I. Sobolev, *Elements of Functional Analysis*, 2nd ed., "Nauka," 1965.
5. P. E. Sobolevskii, V. A. Potapelenko, Siberian Math. Journal, 8, No. 1, 123 (1967).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.