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Abstract

Full Text

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ON SOME LINEAR POLYNOMIAL OPERATIONS IN THE COMPLEX DOMAIN

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1°. Let $\widetilde{C}$ be the space of all continuous 2π-periodic functions $f(x)$ with norm $\|f\|_{\widetilde{C}} = \max|f(x)|$. Let $U_{n,m}(f)$ be a linear operation from $\widetilde{C}$ into $\widetilde{C}$ having the following properties: 1) for any $f \in \widetilde{C}$, $U_{n,m}(f)$ is a trigonometric polynomial of order not exceeding $n+m$, where n and m are positive integers; 2) if f is a trigonometric polynomial of order not exceeding n, then $U_{n,m}(f) = f$. We denote the set of all such $U_{n,m}$ by $\Omega_{n,n+m}$. It is obvious that the Vallée-Poussin sum

$$\sigma_{n,m}(f) = \frac{1}{m+1}\sum_{k=m}^{n+m} s_k(f), \tag{1}$$

where $s_k(f)$ is a partial sum of the Fourier series, belongs to $\Omega_{n,n+m}$. Put

$$\widetilde{U}_{n,m}(f) = \frac{1}{2\pi}\int_0^{2\pi} (U_{n,m}(f_t))_{-t}\,dt,$$

where $f_t(x) = f(x+t)$. In ([1]) it was proved that for any $U_{n,m} \in \Omega_{n,n+m}$ and any $f \in \widetilde{C}$ the equality

$$\widetilde{U}_{n,m}(f) = s_n(f) + \frac{1}{2}\sum_{k=n+1}^{n+m} (A_k \cos kx + B_k \sin kx), \tag{2}$$

holds, where the numbers A_k and B_k are computed by the formulas

$$A_k = a_k(c_k^{(1)} + s_k^{(2)}) + b_k(s_k^{(1)} - c_k^{(2)}), \qquad B_k = a_k(c_k^{(2)} - s_k^{(1)}) + b_k(s_k^{(2)} + c_k^{(1)}).$$

Here a_k and b_k, $c_k^{(1)}$ and $c_k^{(2)}$, $s_k^{(1)}$ and $s_k^{(2)}$ are the Fourier coefficients of order k, respectively, for the functions $f(x)$, $U_{n,m}(\cos kx)$, $U_{n,m}(\sin kx)$. With the aid of formula (2) and its modifications and generalizations, various problems in the theory of linear polynomial operations were solved in the 2π-periodic real case

([2–4]). In the present article the results from ([1]) are extended to the complex domain.

2°. Let D be an arbitrary finite domain with simply connected complement D_1 and rectifiable boundary Γ. By $w = \Phi(z)$ denote the function mapping D_1 conformally onto the domain $|w| \geq 1$ of the w-plane under the condition $\Phi(\infty) = \infty$. Let $z = \psi(w)$ be the inverse function. By $A(D)$ we denote the set of all functions $f(z)$ continuous in the closed domain $\overline{D}$ and analytic inside D. For $f \in A(D)$ we define the norm by the equality $\|f\| = \max_{z\in\overline{D}} |f(z)|$. For real t and $f \in A(D)$ define the operations

$$f_t(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f\{\psi[\Phi(\zeta)e^{it}]\}}{\zeta - z}\, d\zeta; \tag{3}$$

$$f_{t^\wedge}(z) = \frac{f_t(z) + f_{-t}(z)}{2}. \tag{4}$$

We shall call transformation (3) a shift of the function f by l, and transformation (4) an even shift of the function f by t. Operations (3) and (4) were considered in another connection by V. K. Dzyadyk ([5,6]).

Let F_k denote the Faber polynomial of degree k generated by the domain D ([7–9,14]). The equalities

$$(F_k)_t = F_k e^{ikt}, \qquad (F_k)_t \sim \cos kt, \tag{5}$$

are easily proved; they are consequences of an identity of V. K. Dzyadyk ([5,6]).

Let $U_{n,m}$ be a linear operation from $A(D)$ into $A(D)$ having the following properties: 1) for any $f \in A(D)$, $U_{n,m}(f)$ is a polynomial of degree $\leq n + m$; 2) if f is a polynomial of degree $\leq n$, then $U_{n,m}(f) = f$. We shall also denote the set of all such $U_{n,m}$ by $\Omega_{n,n+m}$. The most important example of an operation of type $U_{n,m}$ is furnished by the sum (1), where now

$$s_n(f) = \sum_{k=0}^{n} a_k F_k(z), \tag{6}$$

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f[\psi(e^{it})]\, e^{-ikt}\, dt. \tag{7}$$

Theorem 1. Let $f \in A(D)$ and $U_{n,m} \in \Omega_{n,n+m}$; then

$$\frac{1}{2\pi} \int_0^{2\pi} (U_{n,m}(f_t))_{-t}\, dt = s_n(f) + \sum_{k=n+1}^{n+m} a_k \alpha_{k,k} F_k, \tag{8}$$

where a_k is determined according to (7) and

$$\alpha_{k,j} = \frac{1}{2\pi} \int_0^{2\pi} U_{n,m}(F_k)(\psi(e^{it}))e^{-ijt}\,dt. \tag{9}$$

We outline the proof. Denote the left-hand side of equality (8) by $\widetilde{U}_{n,m}(f)$. One may verify that for any $f \in A(D)$, $\widetilde{U}_{n,m}(f) = \widetilde{U}_{n,m}(s_{n+m}(f))$, where $s_n(f)$ is defined according to (6). It is clear that

$$\widetilde{U}_{n,m}(s_{n+m}(f)) = \widetilde{U}_{n,m}(s_n(f)) + \widetilde{U}_{n,m}(s_{n+m}(f) - s_n(f)). \tag{10}$$

According to (6) and (5), we have

$$(s_n(f))_t = \sum_{k=0}^{n} a_k F_k e^{ikt}.$$

Since $U_{n,m} \in \Omega_{n,n+m}$, by virtue of (5) we obtain

$$U_{n,m}[(s_n(f))_t] = \sum_{k=0}^{n} a_k F_k e^{ikt}.$$

Consequently,

$$\{U_{n,m}[(s_n(f))_t]\}_{-t} = s_n(f).$$

Therefore

$$\widetilde{U}_{n,m}(s_n(f)) = s_n(f). \tag{11}$$

Since

$$U_{n,m}(F_k) = \sum_{j=0}^{n+m} \alpha_{k,j} F_j,$$

where $\{\alpha_{k,j}\}$ are determined according to (9), by virtue of (5) we have

$$\widetilde{U}_{n,m}(F_k) = \alpha_{k,k} F_k. \tag{12}$$

Let us now note that

$$U_{n,m}(s_{n+m}(f) - s_n(f)) = \sum_{k=n+1}^{n+m} a_k \widetilde{U}_{n,m}(F_k).$$

Therefore, from (12) it follows that

$$\widetilde{U}_{n,m}(s_{n+m}(f) - s_n(f)) = \sum_{k=n+1}^{n+m} a_k \alpha_{k,k} F_k. \tag{13}$$

From (10), (11), and (13), (8) follows.

In the same way one proves

Theorem 2. Let $f \in A(D)$ and $U_{n,m} \in \Omega_{n,n+m}$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} [U_{n,m}(f_t^\wedge)]_{t^\wedge}\, dt = \frac{1}{2} s_n(f) + \frac{a_0}{2} + \frac{1}{2}\sum_{j=n+1}^{n+m} a_j \alpha_{j,j} F_j, \tag{14}$$

where $\{a_k\}$ and $\{\alpha_{j,j}\}$ are the same numbers as in Theorem 1.

3°. Put

$$\chi(z) = \sum_{k=m+1}^{n-m} \frac{F_k(z) - F_{2n+2m+2-k}(z)}{n+m+1-k},$$

where F_k is the Faber polynomial of degree k. It is known that, in the trigonometric case, similar polynomials were first considered by Fejér ([10]). Let the equation of the contour Γ of the domain D have the form $\xi = \xi(s)$, where s is the length of the arc measured from some fixed point of the contour. We shall assume that $\xi''(s)$ exists, and moreover $\xi''(s) \in \operatorname{Lip}\alpha$ for some α, $0 < \alpha < 1$. The following holds.

Theorem 3. Let the contour Γ satisfy the stated conditions; then there exists a constant A such that, for all $z \in \overline{D}$, the inequality holds

$$|\chi(z)| \le A.$$

Denote by $A'(D)$ the subset of $A(D)$ consisting of all functions analytic in the closed domain $\overline{D}$. For what follows we need a lemma.

Lemma. Let a function $\psi(w)$, mapping conformally the domain $|w| > 1$ onto the domain D_1, have at all points of the circle $|w| = 1$ a finite continuous nonzero derivative $\psi'(w)$, satisfying a Lipschitz condition. Then the operator (4), considered as a mapping from $A'(D)$ into $A'(D)$, is linear, and for any

$f \in A'(D)$ and all t, $-\infty < t < \infty$, the inequality $\|f_t^\wedge\| \le C\|f\|$ holds, where the constant C depends neither on f nor on t.

4°. **Theorem 4.** For any $U_{n,m} \in \Omega_{n,n+m}$ the inequality holds

$$\|U_{n,m}\| \ge C \ln \frac{n}{m+1}, \tag{15}$$

where the constant $C > 0$ does not depend on n or m.

We outline the proof. By means of general theorems on the possibility of passing to the norm under the integral sign ([11]), and with the aid of the lemma, we obtain that

$$\|U_{n,m}(\chi)\| \ge \frac{1}{C^2} \|\widetilde{U}_{n,m}(\chi)\|. \tag{16}$$

We now estimate $\|U_{n,m}(\chi)\|$. For this we use Theorem 2, taking into account that, in the expansion of the polynomial χ in Faber polynomials, there are no terms with indices k satisfying the inequalities $n+1 \le k \le n+m$, i.e., $a_k = 0$, $n+1 \le k \le n+m$. Therefore equality (14) takes the form

$$\widetilde{U}_{n,m}(\chi, z) = s_n(\chi, z) = \sum_{k=m+1}^{n-m} \frac{F_k(z)}{n+m+1-k}. \tag{17}$$

One can verify that

$$|s_n(\chi, z^*)| \ge C \ln \frac{n}{m+1}, \tag{18}$$

where $C > 0$ is a constant and z^* is the preimage of the point $w = 1$, i.e. $\Phi(z^*) = 1$. From (17) and (18) we infer that

$$|\widetilde{U}_{n,m}(\chi, z^*)| \ge C \ln \frac{n}{m+1}.$$

By virtue of Theorem 3 we have

$$\|\widehat{U}_{n,m}(\chi)\| \ge C \ln \frac{n}{m+1}. \tag{19}$$

From (16) and (19), (15) follows.

The Banach-Steinhaus theorem and Theorem 4 lead to the theorem

Theorem 5. *Whatever the sequence of linear operations* $\{U_{n_i,m_i}\}$, *where* $U_{n_i,m_i} \in \Omega_{n_i,n_i+m_i}$, *with* $\lim_{i\to\infty} \frac{m_i}{n_i} = 0$, *there always exists an* $f \in A(D)$ *such that*

$$\overline{\lim_{i\to\infty}} \|U_{n_i,m_i}(f) - f\| = \infty.$$

Theorems 4 and 5 in the case of the space $\widetilde{C}$ are found in [12].

5°. Let an arbitrary sequence of positive numbers be given, satisfying the inequalities $0 < p_n \leq 1$, $n = 1, 2, ...$, and let $\lim_{n\to\infty} p_n = 1$. The question is whether it is possible to construct a sequence of linear operations $\{U_{n,m}\}$, $U_{n,m} \in \Omega_{n,n+m}$, in such a way that for every $f \in A(D)$ the equality

$$\|U_{n,m}(f) - f\| = O(E_{[np_n]}(f)), \tag{20}$$

holds, where $E_n(f)$ is the best approximation in the domain D of the function $f \in A(D)$ by means of a polynomial of degree n, and $[np_n]$ is the integer part of the number np_n.

Theorem 6. *There does not exist a sequence of operations* $\{U_{n,m}\}$, $U_{n,m} \in \Omega_{n,n+m}$, *satisfying condition (20).*

This theorem for the case of the space $\widetilde{C}$ is found in [13].

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Note: Figure translations are in progress. See original paper for figures.