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ON THE DYNAMIC INSTABILITY OF LINEAR SYSTEMS WITH ALMOST PERIODIC COEFFICIENTS

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Consider a system of linear Hamiltonian equations

$$\mathcal{J}\, dx/dt = [\mathcal{H}_0 + \varepsilon \mathcal{H}(t)]x, \tag{1}$$

Here $x = x(t)$ is an unknown complex n-component vector function of the numerical argument t, $-\infty < t < \infty$; $\mathcal{J}$ is a nonsingular skew-symmetric matrix ($\mathcal{J}^* = -\mathcal{J}$); $\mathcal{H}_0$ is a symmetric positive matrix ($\mathcal{H}_0^* = \mathcal{H}_0 > 0$). The symmetric matrix function $\mathcal{H}(t)$ is a trigonometric polynomial of the form

$$\mathcal{H}(t) = \sum_k \mathcal{H}^{(k)} e^{i(k,\omega)_m t}, \tag{2}$$

where $\omega = (\omega_1, \dots, \omega_m)$ is a real m-component vector, called the frequency vector; k is a vector with integer components and

$$(k, \omega)_m = \sum_{j=1}^{m} k_j \omega_j.$$

The positive quantity ε in equation (1) characterizes the "amplitude of the perturbation" and is assumed to be sufficiently small.

Let $X(t)$ be the matriciant of equation (1), normalized by the condition $X(0) = I$, where I is the identity matrix. Denote by $\rho = \rho(\varepsilon, \omega)$ the greatest exponent of the solutions,

$$\rho = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln \|X(t)\|,$$

where $\|X\|$ is the norm of the matrix X. It is not difficult to show that $\rho(\varepsilon, \omega) \geq 0$.

Definition 1. a) The frequency vector $\omega = \omega_0$ will be called **resonant** if there exists a vector ω_1 such that, in the $(m+1)$-dimensional parameter space $\{\varepsilon, \omega\}$, on the ray $\{\varepsilon, \omega_0 + \varepsilon\omega_1\}$,

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \rho(\varepsilon, \omega_0 + \varepsilon\omega_1) > 0. \tag{3}$$

b) The set of all rays of the indicated form, considered for small ε, will be called the **set of essential instability of solutions of equation (1)** adjoining the point $\{0, \omega_0\}$. c) The set in the space $\{\varepsilon, \omega\}$, defined for a given $\alpha > 0$ by the inequality $\rho(\varepsilon, \omega) > \alpha$, will be called the **set of α-exponential growth of solutions of the equation**.

The problem consists in finding the resonant frequency vectors for equation (1), the corresponding sets of essential instability of solutions, and in estimating the perturbation amplitude necessary for reaching, along a given ray, the set of α-exponential growth of solutions for sufficiently small values of α. Some answers to these questions are given in Theorems 1-2 below.

Similar problems were considered in works ([1–6]) for systems with periodic coefficients; however, the techniques used there are not applicable in the present case. Closely related problems in the case of linear equations (not necessarily Hamiltonian ones) with quasiharmonic coefficients were studied in ([7,8]), where an essential point was the construction of a formal analogue of the Floquet-Lyapunov theorem ([7]).

In the present work, a combination of the averaging method with

by the method of estimating the characteristic numbers of the solutions of the averaged equation with the aid of certain quadratic forms–Lyapunov functions. The first method is widely known from the works ([9], [10]); the second received substantial development in ([11]). We note that the main difficulty in using the averaging method for $\omega = \omega(\varepsilon)$ consists in the appearance, in "critical" (resonance) cases, of small denominators in the transformation carrying one over to the averaged equation. This remark also applies to the technique connected with the use of the formal analogue of the Floquet–Lyapunov theorem. For the Hamiltonian equation considered in the present note, in some cases these difficulties can be overcome. Denote by K the set of vectors k with integer components occurring in the sum (2), and by λ_j the eigenvalues of the matrix $i\mathcal{J}^{-1}\mathcal{H}_0$, numbered so that $j\lambda_j > 0$. The set K is, by assumption, finite.

Theorem 1. *If, for $\omega = \omega_0$, for no vectors $k \in K$ and for no numbers λ_p, λ_r of opposite sign is the relation*

$$(k, \omega_0)_m = \lambda_p - \lambda_r, \tag{4}$$

satisfied, then along any ray $\{\varepsilon, \omega_0 + \varepsilon\omega_1\}$ the estimate $\rho(\varepsilon, \omega_0 + \varepsilon\omega_1) \le C\varepsilon^2$ holds, where C is some constant, uniform with respect to variation of the vector ω_1 in any bounded set.

If relation (4) is satisfied for some vectors $k \in K$ and numbers λ_p, λ_r of opposite sign, then in the class of Hamiltonian equations with matrix functions $\mathcal{H}(t)$ of the form (2) there will be one for which the frequency vector ω_0 is resonant.

Remark. Denote by $\mathfrak{M}$ the class of all Hamiltonian equations (1) with almost-periodic matrix functions $\mathcal{H}(t)$. If the frequency vector ω_0 is resonant for at least one equation of the class $\mathfrak{M}$, then we shall call it **resonant in the class** $\mathfrak{M}$. It follows from Theorem 1 that the set of frequency vectors resonant in the class $\mathfrak{M}$ is everywhere dense in the hyperplane $\{0, \omega\}$.

We give formulas for finding the sets of substantial instability of solutions of equation (1) (see Definition 1) in a special, but important for applications, "general" case. By Theorem 1, resonance vectors in the space $\{\omega\}$ can be located only on hyperplanes of the form (4). Speaking somewhat imprecisely, the "general" case occurs if the vector ω_0 belongs to only one of these hyperplanes. More precisely this means the following.

Definition 2. We shall say that, for $\varepsilon = 0$ and $\omega = \omega_0$, the equations (1) are in the **general case** if: 1) there exists a unique pair of eigenvalues λ_p, λ_r of opposite sign and a unique vector $k \in K$, $k = k_{pr}$, satisfying the relation

$$(k_{pr}, \omega_0) = \lambda_p - \lambda_r, \qquad \lambda_p > 0; \tag{5}$$

2) for all eigenvalues λ_q commensurable with λ_p or λ_r modulo $(K, \omega_0)_m$,* the estimate

$$|\lambda_p - \lambda_q| > \max_{k \in K} |(k, \omega_0)_m|, \qquad |\lambda_r - \lambda_q| > \max_{k \in K} |(k, \omega_0)_m|. \tag{6}$$

Theorem 2. *Suppose that, for the Hamiltonian equation (1), at $\varepsilon = 0$ and for the frequency vector $\omega = \omega_0$, the general case occurs. Then the following assertions are true.*

1. The set of substantial instability of solutions adjoining the point $\{0, \omega_0\}$ has the form of a cone $\{\varepsilon, \omega_0 + \varepsilon\omega_1\}$, where ω_1 is any vector satisfying the inequalities

$$\gamma_{pp} + \gamma_{rr} - 2|\gamma_{pr}| < (k_{pr}, \omega_1)_m < \gamma_{pp} + \gamma_{rr} + 2|\gamma_{pr}|. \tag{7}$$

* The numbers λ_p and λ_q are commensurable modulo $(K, \omega_0)_m$ if in the set K there is a vector k for which λ_p and λ_q are commensurable modulo $(k, \omega_0)_m$.

Here $\gamma_{pp} = (\mathcal{H}^{(0)} a_p, a_p)$; $\gamma_{rr} = (\mathcal{H}^{(0)} a_r, a_r)$; $\gamma_{pr} = (\mathcal{H}^{k_{pr}} a_p, a_r)$; a_p, a_r are eigenvectors of the matrix $i\mathcal{J}^{-1}\mathcal{H}_0$ corresponding to the eigenvalues λ_p, λ_r, normalized by the condition $(\mathcal{J} a_p, a_r) = i \operatorname{sign} p\, \delta_{pr}{}^*$; the integer vector k_{pr} is determined by equality (5), and $\mathcal{H}^{(k_{pr})}$ is the corresponding Fourier component of the function $\mathcal{H}(t)$ in the sum (2).

2. Let $\varepsilon(\alpha, \omega_1)$ denote the amplitude of the perturbation necessary to attain a set of α-exponential growth of solutions along the ray $\{\varepsilon, \omega_0 + \varepsilon\omega_1\}$. If $|\gamma_{pr}| \neq 0$ and the vector ω_1 satisfies inequality (7), then

$$\varepsilon(\alpha, \omega_1) = 2\alpha/\sqrt{4|\gamma_{pr}|^2 - \left[\gamma_{pp} + \gamma_{rr} - (k_{pr}, \omega_1)m\right]^2} + O(\alpha)^2. \tag{8}$$

3. If $\varepsilon(\alpha)$ is the exact upper bound of the values of ε such that in the domain of instability for the general solution $x(t)$ of equation (1) the estimate $(x(t), x(t)) = O(\exp 2\alpha t)$ is valid, then

$$\varepsilon(\alpha) = \alpha/|\gamma_{pr}| + O(\alpha^2). \tag{9}$$

4. Suppose that for equation (1), at $\varepsilon = 0$ and $\omega = \omega_0$, there is no general case, but there exist N pairs of numbers $\lambda_{ps}, \lambda_{rs}$, $s = 1, 2, \dots, N$, of different signs, satisfying relations (5)–(6), and, moreover, among different pairs there are no numbers with identical indices. Then the set of essential instability of the solutions of equation (1) adjacent to the point $\{0, \omega_0\}$ is the union, in the set-theoretic sense, of N cones constructed for each pair $\lambda_{ps}, \lambda_{rs}$ as indicated above in item 1. The quantities $\varepsilon(\alpha, \omega_1)$ and $\varepsilon(\alpha)$ in this case are determined as the maxima over s of the corresponding quantities constructed for each of the N cones.

It follows from Theorem 2 that the larger the quantity

$$|\gamma_{pr}| = |(\mathcal{H}^{(k_{pr})} a_p, a_r)|,$$

the closer to the point $\{0, \omega_0\}$, for small α, the set of α-exponential growth of solutions approaches. Thus, for Hamiltonian systems the quantity $|\gamma_{pr}|$ may be taken as a characteristic of the "danger" of the resonant frequency vector ω_0.

Formulas (7), (9) were obtained in (1) (see also (2-6, 12, 13)) in the study of equations with periodic coefficients.

Example. Consider the equation

$$d^2 f/dt^2 + [P_0 - \varphi(t)N]\, f = 0, \tag{10}$$

to which certain problems of the dynamic theory of plates and of the plane form of buckling lead (14). Here

$$P_0 = \begin{Vmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{Vmatrix}, \qquad 0 < \sigma_1 < \sigma_2, \qquad N = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix},$$

$$\varphi(t) = a + \beta \cos \omega_1 t + \gamma \cos \omega_2 t, \qquad \omega_1 \geq 0, \quad \omega_2 \geq 0.$$

The role of the small parameter is played by the quantity $\varepsilon = \sqrt{a^2 + \beta^2 + \gamma^2}$. By means of the substitution $x = (P_0^{1/4} f,\ P_0^{-1/4} df/dt)$, equation (10) is reduced to equation (1). In the present case $\omega = (\omega_1, \omega_2)$, and the set K consists of the vectors $(0, 0)$; $(1, 0)$; $(0, 1)$; $(-1, 0)$; $(0, -1)$. Consider, in the plane $\{\omega\}$, the lines defined by the equations

$$\omega_1 = 2\sigma_1; \qquad \omega_1 = 2\sigma_2; \qquad \omega_2 = 2\sigma_1; \qquad \omega_2 = 2\sigma_2; \qquad \omega_1 = \sigma_1 + \sigma_2;$$
$$\omega_2 = \sigma_1 + \sigma_2. \tag{11}$$

If the frequency vector $\omega_0 = (\omega_1^{(0)}, \omega_2^{(0)})$ does not lie on any of these lines, then, by Theorem 1, ω_0 is a nonresonant vector. In this case, when the frequency vector ω is varied according to the law $\omega = \omega_0 + \sqrt{a^2+\beta^2+\gamma^2}\,\omega$, where ω is an arbitrary fixed vector, the leading exponent of the solutions of equation (10) is a quantity of order $O(a^2+\beta^2+\gamma^2)$. If the vector $\omega = \omega_0$ lies on only one of the lines (11) and at the same time does not lie on

* Parentheses denote the scalar product of the vectors standing between them; δ_{pr} is the Kronecker symbol: δ

on the lines $\omega_1 = \sigma_2 - \sigma_1$ or $\omega_2 = \sigma_2 - \sigma_1$, then the general case occurs. With the aid of the formulas of Theorem 2 it is not difficult to show that, in the case of the first four straight lines (11), the vector ω_0 will likewise not be resonant (the quantity γ_{pr} appearing in Theorem 2 here becomes zero). Only vectors lying on the straight lines $\omega_1 = \sigma_1 + \sigma_2$, $\omega_2 = \sigma_1 + \sigma_2$ will be resonant. If the vector ω_0 coincides with none of the three vectors
$(\sigma_1+\sigma_2, \sigma_1+\sigma_2)$; $(\sigma_1+\sigma_2, \sigma_2-\sigma_1)$; $(\sigma_2-\sigma_1, \sigma_1+\sigma_2)$, then Theorem 2 makes it possible to determine directly the corresponding sets of essential instability of the solutions of equation (10).

Consider, for example, the resonant frequency vector $\omega_0 = (\sigma_1+\sigma_2, \omega_2^{(0)})$, where $\omega_2^{(0)}$ is an arbitrary positive number; $\omega_2^{(0)} \neq \sigma_1+\sigma_2$, $\omega_2^{(0)} \neq \sigma_2-\sigma_1$. The set of essential instability of the solutions adjoining, for $\alpha = \beta = \gamma = 0$, the vector ω_0, consists of the rays $\{\varepsilon, \omega(\varepsilon)\}$, where $\omega(\varepsilon) = \{\omega_1, \omega_2^{(0)} + \varepsilon\omega_3\}$, ω_3 is an arbitrary number and ω_1 is any number in the interval

$$(\sigma_1+\sigma_2 - |\beta|/2\sqrt{\sigma_1\sigma_2},\ \sigma_1+\sigma_2 + |\beta|/2\sqrt{\sigma_1\sigma_2}).$$

The formulas for the resonant vector $\omega_0 = (\omega_1^{(0)}, \sigma_1+\sigma_2)$, $\omega_1^{(0)} \neq \sigma_1+\sigma_2$, $\omega_1^{(0)} \neq \sigma_2-\sigma_1$, are obtained similarly. Thus, both in the general case and in the approximation under consideration, in system (10) only one resonating harmonic is essential, causing parametric resonance. If the resonance is caused by both harmonics (in the present example this is possible only for
$\omega_0 = (\sigma_1+\sigma_2, \sigma_2-\sigma_1)$; $\omega_0 = (\sigma_2-\sigma_1, \sigma_1+\sigma_2)$; $\omega_0 = (\sigma_1+\sigma_2, \sigma_1+\sigma_2)$), then we have a special case not covered by Theorem 2. For the resonant vectors $\omega_0 = (\sigma_1+\sigma_2, \sigma_2-\sigma_1)$ and $\omega_0 = (\sigma_2-\sigma_1, \sigma_1+\sigma_2)$ the computations are not much more difficult than in the general case. The set of essential instability of the solutions of equation (10), adjoining, for $\alpha = \beta = \gamma = 0$, the vector $\omega_0 = (\sigma_1+\sigma_2, \sigma_2-\sigma_1)$ (respectively, the vector $\omega_0 = (\sigma_2-\sigma_1, \sigma_1+\sigma_2)$), consists of the rays $\{\varepsilon, \omega_0+\delta\}$, where δ is any vector whose components δ_1, δ_2

satisfy the inequality

$$\beta^2/\delta_1^2 - \gamma^2/\delta_2^2 > 4\sigma_1\sigma_2$$
$$(\gamma^2/\delta_1^2 - \beta^2/\delta_2^2 > 4\sigma_1\sigma_2).$$

The formulas obtained show that the interaction of the resonating harmonics of the parametric perturbation in this case leads to a narrowing of the set of essential instability of the solutions in comparison with the general case considered above, i.e., leads to a certain stabilization of the oscillations.

Thus, the structure of the region of essential instability of the solutions remains uninvestigated only in a neighborhood of one point $\{0, \omega_0\}$, where $\omega_0 = (\sigma_1 + \sigma_2, \sigma_1 + \sigma_2)$. This is connected with the appearance of the small denominators mentioned earlier. In all the cases considered above, the method of averaging and its natural modifications made it possible to reduce the original equation to an equation with a principal part independent of t, which made it possible to obtain effective formulas. For the case $\omega_0 = (\sigma_1 + \sigma_2, \sigma_1 + \sigma_2)$ one can only arrange that the principal part be periodic in time. This, however, is insufficient for obtaining the required formulas.

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Note: Figure translations are in progress. See original paper for figures.