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**Abstract**

**Full Text**

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**THE INITIAL PROBLEM FOR GENERAL SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS**

*(Presented by Academician I. G. Petrovskii on 1 VII 1967)*

Considering a general class of parabolic systems, V. A. Solonnikov first noted <sup>(1)</sup> that for such systems the Cauchy problem may prove to be ill-posed, and that it is natural to replace the Cauchy problem by an initial problem associated with the original system of differential equations by an algebraic condition analogous to the condition of Ya. B. Lopatinskii in the theory of elliptic and parabolic boundary-value problems.

In the present note the solvability of the initial problem is proved for a general system of differential equations, unresolved with respect to the highest derivatives in  $t$ , with constant coefficients. In doing so, essential use is made of the method of A. G. Kostyuchenko and G. I. Eskin <sup>(2)</sup>, and of A. Friedman <sup>(3)</sup>, developed by them in the study of the Cauchy problem.

1. For systems with constant coefficients

$$l(\partial/\partial t, i\partial/\partial x)u(x, t) = 0 \tag{1}$$

the initial problem is considered

$$C(\partial/\partial t, i\partial/\partial x)u(x, t)|_{t=0} = u_0(x); \tag{2}$$

$l(\partial/\partial t, i\partial/\partial x)$ ,  $C(\partial/\partial t, i\partial/\partial x)$  are matrix differential operators with elements  $l_{kj}(\partial/\partial t, i\partial/\partial x)$  ( $k, j = 1, \dots, N$ );  $C_{\alpha k}(\partial/\partial t, i\partial/\partial x)$  ( $\alpha = 1, \dots, r$ ;  $k = 1, \dots, N$ );  $r$  is the degree in  $\lambda$  of the polynomial

$$L(\lambda, \sigma) = \det l(\lambda, \sigma) = \sum_{i=0}^r P_i(\sigma)\lambda^{r-i}; \tag{3}$$

$l(\lambda, \sigma)$  is obtained from  $l(\partial/\partial t, i\partial/\partial x)$  by replacing  $\partial/\partial t$  by  $\lambda$  and  $i\partial/\partial x$  by  $\sigma$ ;  $x = (x_1, \dots, x_n)$ .

Denote by  $q$  and  $p$  the highest orders of differentiation in (1) and (2) with respect to  $x$  and  $t$ , respectively.

Assume that the following basic conditions are satisfied:

- 1)  $P_0(\sigma) \neq 0$  for all real  $\sigma$ ;
- 2)  $\Lambda(\sigma) = \max_{1 \leq j \leq z} \operatorname{Re} \lambda_j(\sigma) \leq C$  ( $\lambda_j(\sigma)$  are the roots of the equation  $L(\lambda, \sigma) = 0$ );
- 3) the rows of the matrix  $A(\lambda, \sigma) = C(\lambda, \sigma)\widehat{L}(\lambda, \sigma)$ , as polynomials in  $\lambda$ , are linearly independent modulo the polynomial  $L(\lambda, \sigma)$  for all real  $\sigma$  ( $\widehat{L}(\lambda, \sigma)$  is the matrix reciprocal to  $l(\lambda, \sigma)$ ).

Then the following holds.

**Theorem 1.** *If conditions 1), 2), 3) are satisfied, then there exists a unique solution  $u(x, t)$  of problem (1), (2) in  $\mathcal{H}$  ( ${}^A$ ), p. 215); moreover, if the initial functions  $u_0(x) \in \mathcal{H}(s)$ , then for every  $t$  the solution  $u(x, t) \in \mathcal{H}(s + M)$  ( $M$  is defined in (5)).*

*If  $u_0(x)$ , together with derivatives up to order  $M + \max(p, p_0, q) + n + 1$ , is square summable, then the functions  $u(x, t)$  satisfy (1), (2) in the classical sense and, for every  $t$ ,*

$$D_x^l u(x, t) \in L_2, \quad 0 \leq |l| \leq q.$$

The proof is based on the study of the explicit formula for the solution of the problem for the system of ordinary differential equations obtained by the Fourier transform with respect to  $x$  of (1), (2):

$$l(d/dt, \sigma)\widehat{u}(\sigma, t) = 0, \quad (1')$$

$$C(d/dt, \sigma)\widehat{u}(\sigma, t)|_{t=0} = \widehat{u}_0(\sigma). \quad (2')$$

The solution of problem (1'), (2') is constructed in the same way as in (1), and we represent it in the form

$$\widehat{u}_j(\sigma, t) = \sum_{i=1}^r \widehat{G}_{ji}(\sigma, t)\widehat{u}_{0i}(\sigma), \quad j = 1, \dots, N. \quad (4)$$

The functions  $\widehat{G}_{ji}(\sigma, t)$  are the solution of the problem

$$\sum_{j=1}^N l_{kj}(d/dt, \sigma)\widehat{G}_{ji}(\sigma, t) = 0,$$

$$\sum_{j=1}^N C_{\alpha j}(d/dt, \sigma)\widehat{G}_{ji}(\sigma, t)|_{t=0} = \delta_{\alpha i} \quad (\alpha = 1, \dots, r)$$

and, by virtue of conditions 1), 2), 3), the estimates

$$|\hat{G}_{ji}(\sigma, t)| \leq K(1 + |\sigma|)^M e^{Ct}, \quad (5)$$

hold for them; the constant  $M$  depends on  $p, q, N, r, p_0$ , where  $p_0$  is the degree order of growth, with respect to  $\sigma$ , of the  $\lambda$ -roots of the equation  $L(\lambda, \sigma) = 0$ .

**2.** Here the question of existence of a solution of problem (1), (2) in classes of growing functions is considered. The proof is carried out in the same way as in (2).

The solution of problem (1), (2) in the space of functionals over spaces of type  $S$  (5) is given by formula (4). Using the method of A. Friedman (3), one can prove the existence of a classical solution of problem (1), (2).

As follows from the Zaidman-Tarskii theorem (6) and by virtue of conditions 1), 2), 3), there exists a domain

$$G_\mu = \{s = \sigma + i\tau, |\tau| \leq K_1(1 + |\sigma|)^\mu\},$$

in which the estimates (5) still hold for the functions  $\hat{G}_{ji}(\sigma + i\tau, t)$ , with large constants  $K$  and  $C$ .

If  $\mu < 0$ , then the following holds.

**Theorem 2.** Let  $\mu < 0$  and let conditions 1), 2), 3) be satisfied. If the functions  $u_0(x)$  are continuous together with derivatives up to order

$$M' = M + \max(p \cdot p_0, q) + \nu \quad (\nu \geq n + 2)$$

and satisfy the inequalities

$$|D_x^l u_0(x)| \leq K_2(1 + |x|)^\gamma$$

for  $0 \leq |l| \leq M'$ , where  $\gamma \geq 0$ ,  $\nu + \mu + 1 \leq (\gamma - n)/|\mu|$ , then there exists a classical solution  $u(x, t)$  of problem (1), (2), and

$$|D_x^l u(x, t)| \leq K_3(1 + |x|)^\gamma, \quad 0 \leq |l| \leq q.$$

If, however,  $\mu \geq 0$ , then the domain  $G_\mu$  contains the strip  $|\tau| \leq d$ ; the functions  $G_{ji}(s, t)$  will be analytic in this strip. In this case the following holds.

**Theorem 3.** Let  $\mu \geq 0$  and let conditions 1), 2), 3) be satisfied. Then there exists a classical solution  $u(x, t)$  of problem (1), (2), if the functions  $u_0(x)$  are continuous together with derivatives up to order  $M'$  and satisfy the inequalities

$$|D_x^l u_0(x)| \leq K_4 e^{d'(x)}, \quad 0 \leq |l| \leq M',$$

and for the solution  $u(x, t)$  the inequalities

$$|D_x^l u(x, t)| \leq K_5 e^{d''(x)}, \quad 0 \leq |l| \leq q.$$

**Remark.** If  $P_0(\sigma) = \text{const}$ , and the matrix  $C(\lambda, \sigma)$  is such that the functions  $\hat{G}_{ji}(s, t)$  are integral in  $s$  for each  $t$  (this condition can be formulated algebraically), then, depending on the behavior of  $\lambda$ —the roots of the equation  $L(\lambda, \sigma) = 0$ , one can define, just as in (7), hyperbolic, parabolic, and Petrovskii-correct systems not resolved with respect to the highest derivatives in  $t$ , and for them prove the existence of a solution of problem (1), (2) in the same classes of functions as in (3).

**3.** Let us consider the case where  $P_0(\sigma)$  vanishes for real  $\sigma$ . It is shown in (2) that even if  $u_0(x)$  is a bounded function, in this case a solution of the Cauchy problem does not always exist.

**Theorem 4.** Assume that the following conditions are satisfied:

- a) the real zeros of the polynomial  $P_0(\sigma)$  are situated in a finite part of the real plane;
- b) condition 2) is satisfied, and for sufficiently large  $|\sigma|$  there exists a strip  $|t| \leq d$  such that  $\Lambda(\sigma + i\tau) \leq C$  in this strip;
- c) if  $P_0(\sigma_0) = 0$ , then as  $\xi \rightarrow \sigma_0$ ,  $\Lambda(\xi) \rightarrow -\infty$ ;
- d) condition 3) is satisfied.

Let the initial functions  $u_0(x)$  satisfy the conditions:

- e)  $u_0(x)$  is continuous together with the derivatives up to order  $M'$ , and

$$|D_x^l u_0(x)| \leq C_\varepsilon e^{\varepsilon'|x|^\alpha}, \quad \varepsilon > 0 \text{ arbitrary}, \quad 0 \leq |l| \leq M', \quad \alpha < 1;$$

- f)  $u_0(\sigma)$  has, in a neighborhood of the zeros of the polynomial  $P_0(\sigma)$ , the form  $[P_0(\sigma)]^k v(\sigma)$ , where  $v(\sigma)$  is a functional of function type (the number  $k$  is determined by the behavior of the functions  $\hat{G}_{ji}(\sigma, t)$  in a neighborhood of the zeros of  $P_0(\sigma)$  and depends on  $p, q, p_0, N, r$ ).

Then there exists a function  $u(x, t)$  which satisfies (1) in the classical sense and (2) in the generalized sense, and for  $t > 0$  satisfies the inequalities:

$$|D_x^l u(x, t)| \leq K_\delta e^{\varepsilon'|x|}, \quad 0 \leq |l| \leq q.$$

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### CITED LITERATURE

1. V. A. Solonnikov, *Trudy Mat. Inst. im. V. A. Steklova AN SSSR*, **83** (1965).
2. A. G. Kostyuchenko, G. I. Eskin, *Trudy Moskov. Mat. Obshch.*, **10**, 273 (1961).
3. A. Friedman, *Generalized Functions and Partial Differential Equations*, 1963.
4. G. E. Shilov, *Mathematical Analysis, Second Special Course*, "Nauka," 1965.
5. I. M. Gel' fand, G. E. Shilov, *Spaces of Basic and Generalized Functions*, Moscow, 1958.
6. A. Seidenberg, *Ann. Math. (2)*, **60** (1954).
7. I. M. Gel' fand, G. E. Shilov, *Some Questions in the Theory of Differential Equations*, Moscow, 1958.

*Note: Figure translations are in progress. See original paper for figures.*

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