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Abstract

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MATHEMATICS

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ON EXTREMAL QUASICONFORMAL MAPPINGS WITH A RESTRICTION ON THE CHARACTERISTIC

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In the present note we shall consider extremal quasiconformal mappings generalizing the extremal mappings of Teichmüller (^{1}).

Let S be a closed oriented Riemann surface of genus $g > 1$; let μ_0 be a fixed, henceforth measurable, Beltrami differential (1) on S, satisfying the condition $|\mu_0| \leq M$ almost everywhere, M a constant, and let E be the set of those points on S where $\mu_0 \neq 0$. Suppose that E has positive Lebesgue measure. We introduce for consideration the Banach space $B(S)$ of measurable Beltrami differentials on S with finite norm:

$$\|\mu\| = \text{ess}_S \sup |\mu_0^{-1}\mu|$$

(obviously, $\mu = 0$ almost everywhere on $S - E$). Let $L(S)$ be the Banach space of quadratic differentials (1) on S, summable with weight $|\mu_0|$, with norm

$$\|\varphi\| = \iint_S |\mu_0|\, |\varphi|\, dx\, dy$$

for $\varphi \in L(S)$. It can be proved that every bounded linear functional $\mu(\varphi)$ in $L(S)$ is representable in the form

$$\mu(\varphi) = \iint_S \mu(z)\varphi(z)\, dx\, dy = \langle \mu, \varphi \rangle,$$

$\mu \in B(S)$, $\varphi \in L(S)$. Denote by $A(S)$ the subspace of holomorphic quadratic differentials, and define the subspace $N(S)$ of locally trivial Beltrami differentials by setting

$$N(S) = A(S)^{\perp}.$$

Let α be a homotopy class of homeomorphisms $g : S \to S'$ preserving orientation, S' a Riemann surface, and suppose that there exists a quasiconformal homeomorphism $g_0 \in \alpha$ such that

$$\bar{\partial} g_0 / \partial g_0 \in B(S)$$

($\bar{\partial} = \partial/\partial\bar{z}$ and $\partial = \partial/\partial z$ are the generalized complex differentiation operators in the sense of S. L. Sobolev).

We pose the following extremal problem.

Problem. Among all quasiconformal homeomorphisms of the class α, find those for which the quantity

$$\tau(g) = \mathrm{ess}_S \sup |\bar{\partial} g / \mu_0 \partial g|$$

is least.

Quasiconformal homeomorphisms giving a solution of the problem will be called $|\mu_0|$-extremal.

Theorem 1. *Quasiconformal $|\mu_0|$-extremal homeomorphisms exist. If f is such a homeomorphism, then*

$$\partial f / \partial f = \tau |\mu_0| \, \bar{\varphi} / |\varphi|,$$

where

$$\tau = \inf_{g \in \alpha} \tau(g), \qquad \varphi \in A(S).$$

The proof of the theorem is carried out by the variational method (^{2}) according to the scheme of paper (^{4}). Represent the surfaces S and S' by Fuchsian groups Γ and Γ', acting in the disk $U : |z| < 1$, and fix in U a fundamental polygon P corresponding to the surface S. By $B(\Gamma)$, $N(\Gamma)$, $L(\Gamma)$, and $A(\Gamma)$ denote the spaces of Γ-invariant objects in U corresponding to the spaces $B(S)$, $N(S)$, $L(S)$, and $A(S)$. By a variation of the surface $S = U/\Gamma$, defined with the aid of $\mu \in B(\Gamma)$ and a number ε, $0 < \varepsilon < \|\mu\|_\infty^{-1}$, is meant the quasiconformal automorphism

$$\zeta = H(z; \varepsilon)$$

of the disk U with complex characteristic

$$\varepsilon\mu + O(\varepsilon^2).$$

If the variation is nor-

normalize by the conditions $1 \to 1;\ i \to i;\ -1 \to -1$, then it can be represented in the form (^{4,5})

$$\zeta = z - \frac{\varepsilon}{\pi} \iint_U \left[\frac{\mu(t)}{t - z} + \frac{z^3 \overline{\mu(t)}}{1 - z\bar{t}} \right] d\sigma_t + M(z, \varepsilon) + \omega(z; \varepsilon) = z + \varepsilon h(z) + \omega(z; \varepsilon),$$

where $|\omega(z;\varepsilon)| < C\varepsilon^2$ for $z \in U$, $M(z;\varepsilon)$ is a polynomial uniquely determined by the normalization. If the variation is defined with the aid of $\nu \in N(\Gamma)$ and $\varepsilon > 0$, then $\hat{H}(\gamma z;\varepsilon) = \gamma \hat{H}(z;\varepsilon)$ for $\gamma \in \Gamma$ and ([1,4])

$$\zeta = z - \frac{\varepsilon}{\pi} \iint_U \frac{\nu(t)}{t - z}\, d\sigma_t + O(\varepsilon^2).$$

We shall prove the lower semicontinuity of the functional $\tau(g)$, from which the existence of $|\mu_0|$-extremal mappings will follow. Let $\{g_n(z)\}_1^\infty$ be a minimizing sequence of quasiconformal mappings of class α: $\lim_{n\to\infty} \tau(g_n) = \tau$, which may be assumed to converge uniformly to a function $f(z)$ on each compact set $F \subset U$, if, if necessary, one chooses a subsequence and changes the numbering. Since $g_n(\gamma z) = \gamma' g_n(z)$, $\gamma \in \Gamma$, $\gamma' \in \Gamma'$, it follows that $f(\gamma z) = \gamma' f(z)$, and therefore $f(z) \neq \text{const}$. Let $g_n = u_n + iv_n$, $f = u + iv$. From the equality $|\overline{\partial} g_n| = |\mu_n|\,|\partial g_n|$ we obtain $|\overline{\partial} g_n| \leq \tau(g_n)|\mu_0|\,|\partial g_n|$, and

$$|\operatorname{grad} u_n|^2 + |\operatorname{grad} v_n|^2 \leq 2\frac{1 + \tau(g_n)^2|\mu_0|^2}{1 - \tau(g_n)^2|\mu_0|^2}\left(|\partial g_n|^2 + |\overline{\partial} g_n|^2\right) = \widetilde{J}(u_n, v_n).$$

Using the weak convergence of the generalized derivatives ([4,6]), we find

$$\iint_U \{|\operatorname{grad} u|^2 + |\operatorname{grad} v|^2\}\omega\, dx\, dy \leq \lim_{n\to\infty} \iint_U \{|\operatorname{grad} u_n|^2 + |\operatorname{grad} v_n|^2\}\omega\, dx\, dy,$$

$$\iint_U \widetilde{J}(u, v)\omega\, dx\, dy = \lim_{n\to\infty} \iint_U \widetilde{J}(u_n, v_n)\omega\, dx\, dy$$

for any nonnegative C^1-function ω finite in the disk U. From the equality

$$\iint_U \{|\operatorname{grad} u|^2 + |\operatorname{grad} v|^2\}\omega\, dx\, dy \leq \iint_U \widetilde{J}(u, v)\omega\, dx\, dy,$$

by virtue of the arbitrariness of ω, it follows that

$$|\operatorname{grad} u|^2 + |\operatorname{grad} v|^2 \leq 2\frac{1 + \tau^2|\mu_0|^2}{1 - \tau^2|\mu_0|^2}\left(|\partial f|^2 - |\overline{\partial} f|^2\right),$$

i.e. $\tau(f) = \tau$.

The second assertion of Theorem 1 follows from the following lemmas.

Lemma 1 (S. L. Krushkal' ([4])). Let E_0 be a set of positive Lebesgue measure, $E_0 \subset E$, and let $\mu \in B(S)$ be a Beltrami differential equal to zero on E_0. There exists $\hat{\mu} \in N(S)$ such that $\hat{\mu} = \mu$ on $E - E_0$ and

$$\operatorname{ess}_E \sup |\mu_0^{-1}\hat{\mu}| \le C(E_0)\|\mu\|.$$

Lemma 2. Let $f(z)$ be a $|\mu_0|$-extremal mapping of class α. Then almost everywhere on E

$$|\mu| = |\bar{\partial} f/\partial f| = \tau|\mu_0|.$$

Proof. We may assume that $\tau > 0$ (if $\tau = 0$, then we have a conformal mapping). Suppose on a set $G \subset E$, $\operatorname{mes} G > 0$, we have

$$|\bar{\partial} f/\partial f| < \tau|\mu_0|.$$

Then there exists a closed set $E_0 \subseteq G$, $\operatorname{mes} E_0 > 0$, on which the function $|\bar{\partial} f/\partial f| - \tau|\mu_0|$ is continuous and the inequality

$$|\bar{\partial} f/\partial f| \le \tau_1|\mu_0|$$

holds for some $\tau_1 < \tau$. Varying the surface U/Γ

with the aid of $\nu \in N(\Gamma)$ and $\varepsilon > 0$, compute the characteristic of the mapping $f_0H^{-1}(\zeta)$

$$\mu^* = \frac{\partial f_0 H^{-1}(\zeta)}{\bar{\partial} f_0 H^{-1}(\zeta)} = \frac{\mu - \varepsilon\nu}{1 - \varepsilon\bar{\mu}\nu} \cdot \frac{\partial H}{\bar{\partial} H}; \qquad \tilde{\mu} = \mu^* \frac{\bar{\partial}\bar{H}}{\partial H}. \tag{1}$$

Obviously, $\tilde{\mu} \in B(\Gamma)$, and

$$\tilde{\mu} = \mu - \varepsilon\nu + \varepsilon\bar{\nu}\mu^2 + O(\varepsilon^2). \tag{2}$$

Putting $\theta = \arg\mu$ and $\theta_1 = \arg\nu$, we find

$$|\tilde{\mu}| = |\mu| - \varepsilon|\nu|(1 - |\mu|^2)\cos(\theta - \theta_1) + O(\varepsilon^2).$$

We choose ν so that on $E - E_0$ the inequality

$$|\tilde{\mu}| < \tau|\mu_0| - \varepsilon\eta|\mu_0|$$

is satisfied, where $\eta > 0$ is a constant, and construct, according to Lemma 1, a differential $\hat{\nu} \in N(\Gamma)$ such that $\hat{\nu} = \nu$ on $E - E_0$ and $\|\hat{\nu}\| < \infty$. Then, for sufficiently small ε, we shall have

$$|\mu| = \left|(\mu - \varepsilon\hat{\nu})/(1 - \varepsilon\bar{\hat{\nu}}\mu)\right| < \tau|\mu_0|$$

almost everywhere on E, which contradicts the extremality of the mapping $f(z)$.

Lemma 3. Let $f(z)$, extremal with respect to $|\mu_0|$, be different from a conformal mapping. Then there exists $\varphi \in A(\Gamma)$ such that

$$\mu = \bar{\partial} f/\partial f = \tau|\mu_0|\bar{\varphi}/|\varphi|.$$

Proof. Let

$$\sup_{\|\varphi\|=1,\ \varphi\in A(\Gamma)} |\langle \mu, \varphi\rangle| = k$$

and $k < \tau$,

$$\tau = |\mu_0^{-1}\mu| = \sup_{\|\varphi\|=1,\ \varphi\in L(\Gamma)} |\langle \mu, \varphi\rangle|.$$

The functional $\mu(\varphi) = \langle \mu, \varphi\rangle$, with norm k, defined on the subspace $A(\Gamma)$, by the Hahn–Banach theorem extends, with preservation of the norm, to $L(\Gamma)$. Let $\mu_1(\varphi)$ be one of the extensions of $\mu(\varphi)$, so that $\mu_1(\varphi) = \mu(\varphi)$ for $\varphi \in A(\Gamma)$ and $\|\mu_1(\varphi)\| = k$, and let $\mu_1 \in B(\Gamma)$ be the Beltrami differential corresponding to $\mu_1(\varphi)$. Obviously, $\mu - \mu_1 = \nu \in N(\Gamma)$. Varying U/Γ with the aid of ν and $\varepsilon > 0$, estimate the difference $\Delta = |\langle \tilde{\mu}, \varphi\rangle|^2 - |\langle \mu, \varphi\rangle|^2$, where $\tilde{\mu}$ is defined by formula (1). Using (2), we find

$$\Delta = -2\varepsilon\ \mathrm{Re}\{\overline{\langle \mu, \varphi\rangle}\langle \nu, \varphi\rangle - \overline{\langle \mu, \varphi\rangle}\langle \bar{\nu}\mu^2, \varphi\rangle\} + O(\varepsilon^2).$$

Let C be the unit sphere in $L(\Gamma)$, and $V(\delta)$ the set of those $\varphi \in C$ for which $\tau - \delta \le |\langle \mu, \varphi\rangle| \le \tau$, $\delta < \tau - k$. We may assume that $\langle \mu, \varphi\rangle > 0$ for a given $\varphi \in V(\delta)$ (if necessary, one can always replace φ by $e^{i\beta}\varphi$, where β is a suitable number). Let $\varphi \in V(\delta)$ be such that $\langle \mu, \varphi\rangle = \tau - \delta_1$, $\delta_1 \le \delta$. Then

$$\Delta = -2\varepsilon(\tau - \delta_1)\,[\mathrm{Re}\langle(1 - \tau^2|\mu_0|^2)\mu, \varphi\rangle - \mathrm{Re}\langle(1 - \tau^2|\mu_0|^2)\mu_1, \varphi\rangle -$$

$$- \mathrm{Re}\langle \bar{\nu}\mu^2 - \tau^2|\mu_0|^2\nu, \varphi\rangle] + O(\varepsilon^2) = -2\varepsilon(\tau - \delta_1)[I_1 - I_2 - I_3] + O(\varepsilon^2).$$

Putting $\theta = \arg\mu$, $\lambda = \arg\varphi$ and observing that

$$|I_2| \le k \iint_E (1 - \tau^2|\mu_0|)|\mu_0|\,|\varphi|\,dx\,dy,$$

we estimate the difference between I_1 and

$$I_4 = (\tau - \delta_1) \iint_E (1 - \tau^2 |\mu_0|^2) |\mu_0| \, |\varphi| \, dx \, dy :$$

$$|I_1 - I_4| = \left| \iint_E |\mu_0|^3 |\varphi| \left(\tau^2 \delta_1 - 2\tau^3 \sin^2 \frac{\theta + \lambda}{2} \right) dx \, dy \right| \leq$$

$$\leq \delta_1 \iint_E |\mu_0| \, |\varphi| \, dx \, dy + 2\tau \iint_E |\mu_0| \, |\varphi| \sin^2 \frac{\theta + \lambda}{2} \, dx \, dy = 2\delta_1).$$

The expression I_3 can be transformed to the form

$$I_3 = 2\tau^2 \operatorname{Im} \iint_E |\mu_0|^2 |\varphi| \, |\mu_1| e^{-i\theta} \sin(\theta + \lambda) \, dx \, dy,$$

and, with the aid of the Cauchy–Schwarz inequality, one obtains

$$|I_3| \leq 4\tau k \iint_E |\mu_0|^3 |\varphi| \left| \sin \frac{\theta + \lambda}{2} \right| dx \, dy \leq$$

$$\leq 4\tau^2 k \left(\iint_E |\mu_0|^5 |\varphi| \sin^2 \frac{\theta + \lambda}{2} \, dx \, dy \right)^{1/2} \leq 2k\sqrt{2\delta\tau^{-1}}.$$

Thus,

$$|\Delta| \geq 2\varepsilon(\tau - \delta_1)[(\tau - \delta - k) \operatorname{ess}_E \inf(1 - \tau^2 |\mu_0|^2) -$$

$$-2\delta - 2k\sqrt{2\delta\tau^{-1}}] + O(\varepsilon^2).$$

Choose δ so small that the expression in square brackets is positive. Then $\Delta < 0$ and $\Delta = O(\varepsilon)$ for any $\varphi \in V(\delta)$. For $\varphi \in V(\delta)$ we have $|\langle \mu, \varphi \rangle| < \tau - \delta + O(\varepsilon)$. For sufficiently small $\varepsilon > 0$ we obtain

$$\tau - \sup_{\varphi \in C} |\langle \mu, \varphi \rangle| = O(\varepsilon),$$

i.e. $\|\mu^*\| < \tau$, which contradicts the extremality of $f(z)$.

The problem of extremal quasiconformal mappings with a restriction on the characteristic was posed by L. I. Volkovyskii (^{3}). For the case of an annulus this problem was solved in (^{8}); another approach, based on the method of extremal metrics, was considered in (^{7}).

Let $p_0(z)$ be a measurable function on S satisfying almost everywhere the inequality $1 \leq p_0(z) \leq Q$, $Q = \text{const}$. The Teichmüller surface $[S', \alpha]$ is the

Riemann surface S', equipped with a unique homotopy class α of orientation-preserving homeomorphisms $S \to S'$. Consider the sets of Teichmüller surfaces that can be obtained under quasiconformal mappings of the surface S with characteristics $\{p(z), \theta(z)\}$ in the sense of M. A. Lavrent' ev, satisfying one of the conditions: a) $p(z) < p_0(z)$; b) $p(z) \leq p_0(z)$; c) $p(z) = p_0(z)$ almost everywhere. By U_1, U_2, U_3 we denote the sets of representative points in the Teichmüller space $T_g(S)$ (^{1}), arising respectively under the restrictions a), b), c).

Using Theorem 1 and some facts from (^{1}), one can prove the following theorem.

Theorem 2. *The set* U_1 *is a bounded simply connected domain in* $T_g(S)$*, and* $\overline{U}_1 = U_2 = U_3$*, where* $\overline{U}_1$ *is the closure of* U_1*.*

Theorem 2 gives a solution of Problem 3 from (^{3}).

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Note: Figure translations are in progress. See original paper for figures.

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