ON THE NONEXISTENCE OF MAPPINGS OF CLASS \(BL^{n/2}\) OF A BALL ONTO A DOMAIN

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Abstract

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ON THE NONEXISTENCE OF MAPPINGS OF CLASS $BL^{n/2}$ OF A BALL ONTO A DOMAIN

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1. A continuous vector function $y = f(x)$, defined in a domain D of n-dimensional Euclidean space E^n and with values in E^m, belongs to the class of functions $BL_K^{n/2}$ if

$$I(f, D) = \int_D \left[\sum_{i=1}^{m} \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j} \right)^2 \right]^{n/2} dx \leqslant K < \infty,$$

where dx is the volume element in E^n, and the derivatives are understood in the sense of S. L. Sobolev.

If $f \in BL_K^{n/2}$ for some K, then $f \in BL^{n/2}$. If the function $y = f(x)$ realizes a homeomorphism of the ball $D : |x| < R$ onto a domain Δ, with $f, f^{-1} \in BL_K^{n/2}$, the domain Δ contains the ball $D_1 : |y| < \delta$, and $f(0) = 0$, then the function f belongs to the class of functions $[BL]_K^{n/2}$. If $f \in [BL]_K^{n/2}$ for some K, then $f \in [BL]^{n/2}$.

For $n = m = 1$, the integral $I(f, D)$ gives the variation of the function f on the interval D. For $n = m = 2$, $I(f, D)$ is the usual Dirichlet integral, equal for conformal mappings to twice the measure of the domain $f(D)$. For quasiconformal mappings with bounded measure of the domain $f(D)$, the integral $I(f, D)$ is bounded.

Of fundamental importance in studying the properties of mappings of class $BL^{n/2}$ is the inequality of Theorem 2 of the work (^{1}). If, instead of $I(f, D)$, one substitutes into this inequality the integral

$$V_f(D) = \int_D \lambda_f^n(x)\, dx,$$

where

$$\lambda_f(x) = \overline{\lim_{\Delta x \to 0}} \frac{|f(x+\Delta x) - f(x)|}{|\Delta x|},$$

then it is preserved up to a constant factor, since

$$\lambda_f(x) \leqslant \left[\sum_{i=1}^{m} \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j}\right)^2\right]^{1/2} \leqslant \sqrt{n}\lambda_f(x).$$

The quantity $V_f(D)$ is naturally called the variation of the function over the domain D. The class of functions $BL_K^{n/2}$ is thus a generalization of the class of functions of bounded variation (more precisely, of the class of functions of bounded variation, absolutely continuous inside the interval D). The classes of functions $BL_K^{n/2}$ and $[BL]_K^{n/2}$ have many properties in common with the class of conformal mappings in the plane. The most essential difference is the absence of an analogue of the Riemann theorem for $n \geqslant 3$. The results given below show that, in the general case for $n \geqslant 3$, the question of the existence of mappings of a ball onto a domain within the class under consideration is of exceptional difficulty. For $n = m = 2$, mappings of this class were studied by J. Lelong-Ferrand (^{2}) and G. D. Suvorov (^{3}). The study of the spatial case for $n = m = 3$ was initiated in the works ($^{4-7}$), and was subsequently continued for $n \geqslant 2$ and arbitrary m in (^{1}, ^{8}, ^{9}).

Below we shall use the following notation: $\rho(M_1, M_2)$ is the distance between sets in E^m; $|x' - x''|$ is the distance between points-

in E^n; $\overline{M}$ is the closure of the set M in E^n; $\partial\Delta$ is the boundary of the domain Δ; $d(M)$ is the diameter of the set M in E^n; $C_D(f, a, l)$ is the cluster set of the function f at the point a of the boundary of the domain D relative to the curve $l \subset D$, i.e., it is the set of limit points of sequences $\{f(a_m)\}$, $a_m \in l$ $(m = 1, 2, \dots)$, $\lim_{m\to\infty} a_m = a$.

2. For bounded domains $\Delta \subset E^n$, compact with respect to the relative distance (^{5}) and belonging to the class of domains $A_1(S)$ (for any sphere S in E^n, every component of the set $S \cap \Delta$ divides the domain Δ into two subdomains), the compactification of the domain by prime ends, analogous to the prime ends of Carathéodory (^{10}), is a compactum (^{8}). The prime ends of this class of domains contain 5 types of prime ends. To the 4 types in Carathéodory' s classification there is added a prime end of the 5th type, whose body has a disconnected closed set of principal points.* As the example of domains given below shows, for domains in E^n $(n \geq 3)$ all 5 types of prime ends are realized. On the other hand, by the corollary of Theorem 1, by means of a homeomorphism of class $BL^{n/2}$ a ball cannot be mapped onto a domain having prime ends of the 5th type.

Theorem 1. *If the function $y = f(x)$ realizes a homeomorphic mapping of class $BL^{n/2}$ of the ball $D:\ |x| < 1$ onto a bounded domain Δ, then for every non-tangential curve $l \subset D$ tending to a point $a \in \partial D$, the set $C_D(f, a, l)$ coincides with the set $|e|_1$ of principal points of the prime end $e = f(a)$.*

This theorem is an analogue of the well-known Lindelöf theorem for conformal mappings. For mappings of class BL it was established by J. Lelong-Ferrand (2). The proof of this theorem is based on the inequality from (1) and the results from (8).

Corollary. *There do not exist homeomorphic mappings of class $BL^{n/2}$ of the ball $D:\ |x| < 1$ onto a bounded domain Δ having prime ends with a disconnected set of principal points.*

We give an example of domains in E^3, homeomorphic to the ball and compact with respect to the relative distance, which contain prime ends of the 5th type.

Example 1. Consider in E^3 the cube $R = \{x \in E^3 :\ 0 < x_i < 1\ (i = 1, 2, 3)\}$ and the sets $T_{mp} = \{x \in R :\ x_1 = 1/2^m 3^p\}$ $(m, p = 1, 2, ...)$. Let F be an arbitrary closed set on the face $T_0 = \{x \in \overline{R} :\ x_1 = 0\}$ of the cube R. We construct a domain Δ having a prime end e such that $|e| = T_0,\ |e|_1 = F$. We shall assume that F is an infinite set. For finite F, the constructions are carried out analogously.

Let $\{a_m\}$ $(m = 1, 2, ...)$, $\{a_m\} \subset F$, be some countable set of points of the face T_0, everywhere dense in F, with the points of $\{a_m\}$ not lying on the edges of the cube R. Put $Q_{mp} = \{x \in T_{mp} :\ |b_{mp} - x| < r_{mp}\}$, where $b_{mp} = a_m + c_{mp}$, $c_{mp} = (1/2^m 3^p, 0, 0)$, $r_{mp} = \min[1/2^m 3^p, \rho(a_m, \partial T_0)]$, and ∂T_0 is the boundary of T_0, considered as a plane set (in the plane $x_1 = 0$). Then the set

$$\Delta = R \setminus \left[\bigcup_{m,p=1}^{\infty} (T_{mp} \setminus Q_{mp}) \right]$$

is a bounded domain that is homeomorphic to the ball and compact with respect to the relative distance. Consider a sequence of sections $\{q_n\}$ $(n = 1, 2, ...)$, whose set coincides with the set of circles $\{Q_{mp}\}$ $(m, p = 1, 2, ...)$ and which are numbered in the order of approach to the face T_0. This sequence of sections defines a prime end e of the domain Δ such that $|e| = T_0,\ |e|_1 = F$. Thus, if

* Let a prime end e of a domain Δ be defined by means of a chain of subdomains $\{g_m\}$. The body of the prime end e is the set $|e| = \bigcap_{m=1}^{\infty} \overline{g}_m$. A point $a \in |e|$ is called a principal point of the prime end e if there exists a chain of sections $\{q_m\}$ defining the prime end e and contracting to the point a.

If F is a disconnected set, then the simple end e of the constructed domain Δ will be of type 5. For $n = 2$ the set of principal points of a simple end is always connected (10).

3. Let us give a lower estimate for the quantity K for a mapping $y = f(x)$ of class $[BL]_K^{n/2}$ of the ball D onto the domain Δ. This estimate will involve the function $\psi(a, \Delta)$ of the variable a, which is defined as follows. Let $G_a^0(\Delta)$ be the component of the set

   $$G_a(\Delta) = \{x \in \Delta : \rho(x, \partial\Delta) > a\},$$

   containing the point 0. Consider a point $a \in \Delta \setminus G_a^0(\Delta)$ and define the function $h(a, a)$ of the variable a: $h(a, a) = \inf d(K)$, where the infimum is taken over all continua $K \subset \Delta$ separating the points a and 0 in Δ and such that the sets $K \cap G_a^0(\Delta)$ and $K \cap \partial\Delta$ are nonempty. Then $\psi(a, \Delta) = \sup h(a, a)$, where the supremum is taken over all points $a \in \Delta \setminus G_a^0(\Delta)$.

Theorem 2. Let $f(x) \in [BL]_K^{n/2}$. Then the following lower estimate for the quantity K holds:

$$K \geq \sup_{0<a<\delta} \min\left\{ \frac{\psi^n(a, \Delta)}{M_n n + \psi^n(a, \Delta)} \left[\ln \frac{R^n}{M_n} + \ln\ln \frac{\delta}{a} \right], \frac{R^n}{M_n} \ln \frac{\delta}{a} \right\}, \tag{1}$$

where M_n is an absolute constant (1).

The proof is based on the inequality of Theorem 2 from (1).

Corollary. If

$$\overline{\lim_{a \to 0}}\, \psi(a, \Delta) \sqrt[n]{\ln\ln \frac{\delta}{a}} = \infty, \tag{2}$$

then there is no mapping $y = f(x)$ of class $[BL]^{n/2}$ of the ball D onto the domain Δ.

Remark. Let $y = f(x)$ be a Q-quasiconformal mapping of the ball $D : |x| < R$ onto the domain Δ of finite measure $m\Delta$, and let $f(0) = 0$. Then $f(x) \in [BL]_K^{n/2}$, where $K \leq n^{n/2} Q^{n-1} \max(mD, m\Delta)$, and hence

$$Q \geq [n^{n/2} \max(mD, m\Delta)]^{1/(n-1)} K^{1/(n-1)}. \tag{3}$$

Inequalities (1) and (3) give a lower estimate for the quasiconformality coefficient Q in terms of the structure of the domain Δ and its boundary, which is expressed by means of the function $\psi(a, \Delta)$.

The corollary of Theorem 2 makes it possible to construct various domains $\Delta \subset E^n$ ($n \geq 3$), homeomorphic to a ball, onto which the ball cannot be mapped by means of functions of class $[BL]^{n/2}$, nor by quasiconformal mappings. It also makes it possible to construct sequences of domains $\{\Delta_m\}$ such that there exist mappings of the ball $\{f_m\}$, $f_m(D) = \Delta_m$, $f_m \in [BL]^{n/2}$ ($m = 1, 2, \ldots$), but the relation $f_m \in [BL]^K$ is not satisfied for any K.

Example 2. Consider in E^3 the cube

$$R = \{x \in E^3 : |x_i| < 1 \ (i = 1, 2, 3)\}.$$

Let $\{a_m\}$ be a sequence of positive numbers $0 < a_m < 1/3$ $(m = 1, 2, \ldots)$, monotonically tending to 0. Put

$$B_m = \{x \in E^3 : |x_1| < b_m,\ a_{2m-1} < x_2 < a_{2m},\ 1 \le x_3 < b_m\},$$

where

$$b_m = \{\ln[-\ln(a_{2m} - a_{2m-1})]\}^{-1/6}.$$

The set $\bar{\Delta}$,

$$\Delta = R \cup \left(\bigcup_{m=1}^{\infty} B_m \right),$$

is a domain homeomorphic to the closed ball; moreover, for this domain relation (2) holds, and hence there is no mapping of class $[BL]^{3/2}$ of the ball onto this domain.

Example 3. Consider the cube

$$R = \{x \in E^3 : |x_i| < 1\ (i = 1, 2, 3)\}.$$

Let $\{a_m\}$ be a sequence of positive numbers $2/3 \le a_m < 1$ $(m = 1, 2, \ldots)$, monotonically tending to 1. Put

$$C_m = \{x \in R : x_2 = a_m\}, \qquad q_m = \{x \in C_m : |x_1| < a_{m+1} - a_m,\ |x_3| < b_m\},$$

where

$$b_m = \frac{1}{2} \exp[-\exp(a_{m+1} - a_m)^{-6}].$$

Then the set

$$\Delta = R \setminus \left[\bigcup_{m=1}^{\infty} (C_m \setminus q_m) \right]$$

is a domain, homeomorphic to a ball and compact in the relative distance (it is constructed according to the type of the domains of Example 1). The domain Δ has

a simple end of the second type, determined by the sequence of sections $\{q_m\}$, and for it relation (2) is satisfied.

Example 4. Consider a domain having the form of a wedge with zero angle $\Delta = \{x \in E^3 :\ |x_1| < 1,\ |x_2| < 1,\ |x_3| < \varphi(x_2)\}$, where $\varphi(x_2) = \exp[-\exp(x_2 + 1)^{-6}]$. It is easy to see that for this domain relation (2) is satisfied. A mapping of class $[BL]^{n/2}$ onto a domain of this form will be impossible also in the case when, as the function $\varphi(x_2)$, one takes one having a higher order of contact with the axis Ox_2 in comparison with the function chosen in this example.

We note that, for quasiconformal mappings, as F. W. Gehring and J. Väisälä established [11], a mapping onto a domain having a ridge directed outward, or a peak directed inward, is impossible. One can give examples of mappings showing that, for mappings of class $[BL]^{n/2}$, such mappings are possible if the ridge and the peak are not too "sharp."

Example 5. Put $\Delta = \{x \in E^3 : \ |x_i| < 1 \ (i = 1, 2, 3)\}$, $R_n = \{x \in \Delta : \ x_2 = 1 - 1/2n, \ -1 < x_3 \leq 0\}$ $(n = 1, 2, ...)$, and consider the sequence of domains $\{\Delta_n\}$, $\Delta_n = \Delta \setminus R_n$. Then there exists no number K and no sequence of mappings $\{f_n(x)\}$ of the ball D onto the domains Δ_n, $f_n(D) = \Delta_n$, such that $f_n(x) \in [BL]_K^{3/2}$ for every n.

Example 6. Put $\Delta = \{x \in E^3 : \ |x_1| < 1, \ |x_2| < 1, \ -1 < x_3 < 2\}$, $R'_n = \{x \in \Delta : \ x_2 \in [-1/2n, 1/2n], \ x_3 = 1\}$, $\Delta_n = \Delta \setminus R_n$. Then there exists no sequence of mappings $\{f_n(x)\}$ of the ball D onto the domains Δ_n, $f_n(D) = \Delta_n$, and no number K such that $f_n \in [BL]_K^{3/2}$ for all n.

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REFERENCES

1. I. S. Ovchinnikov, *Tr. Tomsk. Gos. Univ.*, **189**, 86 (1966).
2. J. Lelong-Ferrand, *Representation conforme et transformations a integrale de Dirichlet bornée*, Paris, 1955.
3. G. D. Suvorov, *Families of Plane Topological Mappings*, Novosibirsk, 1965.
4. I. S. Ovchinnikov, G. D. Suvorov, *DAN*, **154**, No. 3, 523 (1965).
5. I. S. Ovchinnikov, G. D. Suvorov, *Sibirsk. Mat. Zh.*, **6**, No. 6, 1292 (1965).
6. I. S. Ovchinnikov, *DAN*, **161**, No. 3, 526 (1965).
7. I. S. Ovchinnikov, *Tr. Tomsk. Gos. Univ.*, **183**, 32 (1965).
8. I. S. Ovchinnikov, *Tr. Tomsk. Gos. Univ.*, **189**, 96 (1966).
9. I. S. Ovchinnikov, *Some Metric Properties of Spatial Mappings*, Candidate' s dissertation, Tomsk, 1966.
10. C. Caratheodory, *Math. Ann.*, **73**, 323 (1913).
11. F. W. Gehring, J. Väisälä, *Acta mathematica*, **114** (1965); *Collected Translations. Mathematics*, **10**, 6, 60 (1966).

Note: Figure translations are in progress. See original paper for figures.