PROBLEMS OF FINITE CONTROL OF LINEAR SYSTEMS WITH LUMPED PARAMETERS

CYBERNETICS AND CONTROL THEORY

1968

SovietRxiv

Abstract

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UDC 62.501.12

CYBERNETICS AND CONTROL THEORY

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PROBLEMS OF FINITE CONTROL OF LINEAR SYSTEMS WITH LUMPED PARAMETERS

(Presented by Academician B. N. Petrov, June 24, 1967)

1. Formulation of the problem. In the theory of automatic control the following problem is of great importance. For a certain system (object), find controlling actions from a specified class of admissible ones that would transfer the system from a given initial state to another prescribed state in a fixed specified time.

The controlled system is described by differential equations of the form

$$\dot{x} = Ax + Bu + \Phi, \tag{1}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} \dots a_{1n} \\ \vdots \quad \vdots \\ a_{n1} \dots a_{nn} \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{pmatrix};$$

x is a vector in the n-dimensional phase (generally speaking, complex) space X of the system, characterizing the state of the controlled object at each instant of time t, i.e. $x = x(t)$, $0 \le t \le T$, and t is a real independent variable; A is a square $n \times n$ matrix with constant (in the general case complex) elements a_{ij}, $i, j = 1, 2, \dots, n$; B is a vector with constant (possibly complex) coordinates B_i, $i = 1, 2, \dots, n$; $\Phi = \Phi(t)$ is a vector function specified on $0 \le t \le T$; finally, $u = u(t)$ is a scalar controlling parameter, or simply the control. The problem is easily generalized to the case of several control actions, when $u(t) = (u_1(t), \dots, u_r(t))$.

Let a certain class K of admissible controls be given. At first we shall assume that $K = L_2[0, T]$ and also $\Phi(t) \in L_2[0, T]$.

It is required to find a control $u = u(t) \in K$ such that the representative point $x(t)$ of system (1), moving from the prescribed point $x(0) = x^0$, coincides with another point moving in the space X according to a certain law $x^* = x^*(t)$ at the specified instant of time t, i.e. the condition $x(T) = x^*(T)$ must be satisfied. For simplicity of exposition we shall assume that $x^*(T) = 0$ and, consequently, $x(T) = 0$, since otherwise by the substitution of functions in system (1), $w(t) = x(t) - x^*(t)$, the problem can be reduced to the control of a system of type (1), but with zero terminal condition. We shall consider the control $u(t)$ and the function $\Phi(t)$ to be finite functions on the interval $[0, T]$, i.e. identically equal to zero outside the interval $[0, T]$, and the function $x(t)$ a priori to be equal to zero for $t > 0$. If it is possible to find the required finite control, then, together with the functions $u(t)$ and $\Phi(t)$, the function $x(t)$ may also be regarded as finite, i.e. $x(t) \equiv 0$ for $t \notin [0, T]$.

The process of controlling object (1) by means of such a function $u(t)$ on the time interval $[0, T]$ will be called a **finite process**,

and the control itself $u(t)$ will be called a **finite control** on $[0, T]$, or simply a finite control. Thus, the problem is to find a finite control $u(t) \in L_2[0, T]$ that transfers system (1) from the point x^0 to the point 0 in the prescribed time T.

2. Method of solution. Let $u(t)$ be the sought finite control on $[0, T]$, and let $x(t)$ be the corresponding sought finite trajectory. Taking into account the finiteness of the functions $x(t)$, $u(t)$, and $\Phi(t)$, we apply the Fourier transform to system (1). We obtain

$$j\omega\tilde{x}(\omega) - x^0 = A\tilde{x}(\omega) + B\tilde{u}(\omega) + \tilde{\Phi}(\omega), \tag{2}$$

where $\tilde{x}(\omega)$, $\tilde{u}(\omega)$, and $\tilde{\Phi}(\omega)$ are the Fourier transforms of the functions $x(t)$, $u(t)$, and $\Phi(t)$, respectively. The direct and inverse Fourier transforms of a certain function $f(t)$ have the form

$$\tilde{f}(\omega) = F[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt, \qquad f(t) = F^{-1}[\tilde{f}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{j\omega t}\,d\omega. \tag{3}$$

Solving equation (2) with respect to $\tilde{x}(\omega)$, we obtain

$$\tilde{x}(\omega) = \frac{C(\omega)\,[B\tilde{u}(\omega) + x^0 + \tilde{\Phi}(\omega)]}{|j\omega E - A|}, \tag{4}$$

where $|j\omega E - A|$ is the characteristic determinant of system (1) in the variable ω, and $C(\omega)$ is the adjugate matrix of the matrix A ([2]). Since $x(t)$, $u(t)$, and $\Phi(t)$ are, by assumption, finite functions on $[0, T]$, it follows from the Wiener–Paley theorem ([1], [3]) that $\tilde{x}(z)$, $\tilde{u}(z)$, and $\tilde{\Phi}(z)$ are entire functions of the complex variable $z = \omega + j\xi$ of degree T.

It can be shown that the following converse assertion underlies the solution of the finite-control problem.

Theorem. *Let the function $\tilde{u}(\omega)$ have the following properties: 1) $\tilde{u}(\omega) \in L_2[-\infty, \infty]$; 2) $\tilde{u}(\omega)$ can be continued to the entire plane of the complex variable $z = \omega + j\xi$ as an entire function of finite degree T; 3) the inverse Fourier transform of the function $\tilde{u}(\omega)$ is equal to zero for $t < 0$; 4) the function $\tilde{x}(\omega)$ computed in accordance with formula (4) is an entire function.*

Then $u(t) = F^{-1}[\tilde{u}(\omega)]$ is the sought finite control on $[0, T]$, belonging to $L_2[0, T]$, and $x(t) = F^{-1}[\tilde{x}(\omega)]$ is the sought finite process with conditions $x(0) = x^0$, $x(t) = 0$. If system (1) is considered only in a real phase space and contains only real coefficients, then, in order to determine the sought finite control in this case, one must take the real parts of the inverse Fourier transforms of $\tilde{u}(\omega)$ and $\tilde{x}(\omega)$, i.e., the sought finite control and process are respectively equal to

$$u(t) = \operatorname{Re} F^{-1}[\tilde{u}(\omega)], \qquad x(t) = \operatorname{Re} F^{-1}[\tilde{x}(\omega)].$$

Thus, the stated theorem reduces the original problem to determining some entire function $\tilde{u}(z)$ of degree T for which the function $\tilde{x}(z)$, computed by formula (4), is also an entire function ([1]). The proof of this theorem is simple and is based on application of the Wiener–Paley theorem ([3]).

3. Case of simple roots. Suppose first that all roots of the characteristic polynomial $|jzE - A|$ are simple and equal to $z_1, z_2, \dots, z_n$. Then, in order that $\tilde{x}(z)$ be an entire function, it is necessary that the roots $z_1, z_2, \dots, z_n$ be zeros of the numerator in (4), i.e., the conditions

$$C(z_i)\,[B\tilde{u}(z_i) + x^0 + \tilde{\Phi}(z_i)] = 0, \qquad i = 1, 2, \dots, n. \tag{5}$$

Hence

$$\tilde{u}(z_i) = -C(z_i)[x^0 + \tilde{\Phi}(z_i)]/C(z_i)B = \beta_i, \qquad i = 1, 2, \dots, n, \tag{6}$$

where β_i, $i = 1, \dots, n$, are definite numbers.

Thus, the finite-control problem has been reduced to a certain interpolation problem[4]: to find an entire function of finite degree T which, at the n prescribed points $z_1, z_2, \dots, z_n$ (the roots of the characteristic polynomial), assumes the n prescribed values $\beta_1, \beta_2, \dots, \beta_n$ computed by formula (6). Incidentally, we note that for the interpolation problem (6) to be solvable it is necessary and sufficient that all β_i be finite numbers, i.e., that $C(z_i)B \neq 0$ for $i = 1, 2, \dots, n$. It is easy to show that this is the necessary and sufficient controllability condition for system (1), and is equivalent to the condition of linear independence of the vectors[5] $B, AB, A^2B, \dots, A^{n-1}B$.

The solution of this problem is far from unique. The entire set of solutions of this interpolation problem, and consequently of the finite-control problem as well, is given by the Lagrange formula

$$\tilde{u}(\omega) = \sum_{i=1}^{n} \frac{\beta_i \varphi(\omega)}{\varphi'(z_i)(\omega - z_i)} + \gamma(\omega)\varphi(\omega), \tag{7}$$

where $\varphi(\omega)$ is an entire function of degree T, bounded on the real axis ω and vanishing at all points $z_1, z_2, \ldots, z_n$, with $\varphi(z_i) \neq 0,\ i = 1, 2, \ldots, n$; $\gamma(\omega)\varphi(\omega) \in L_2[-\infty, \infty]$ and has finite degree, and also, together with $\varphi(\omega)$, has inverse Fourier transforms which vanish for $t < 0$.

As $\varphi(\omega)$, one may take, for example, a function of the form (for $\gamma(\omega) \equiv 0$)

$$\varphi(\omega) = \prod_{i=1}^{n} \left(1 - e^{-j(\omega - z_i)\tau}\right), \qquad \text{where } T = n\tau. \tag{8}$$

Thus, the desired finite control is the real part of the inverse Fourier transform of the function $\tilde{u}(\omega)$ computed by formula (7).

4. The case of multiple roots. In the case where the root z_k has multiplicity $\alpha_k,\ k = 1, 2, \ldots, s,\ \sum_{k=1}^{s} \alpha_k = n$, the interpolation problem corresponding to the desired finite control has the form

$$\tilde{u}^{(h)}(z_k) = -[C(\omega)[x^0 + \Phi(\omega)]/C(\omega)B]_{\omega = z_k} = \beta_k^h,$$

$$h = 0, 1, \ldots, \alpha_k - 1; \qquad k = 1, 2, \ldots, s. \tag{9}$$

Let $\varphi(\omega)$ be an entire function of degree T, whose inverse Fourier transform is equal to zero for $t < 0$. In addition, $\varphi(\omega)$ is bounded as $|\omega| \to \infty$, and there exist finite derivatives

$$[(\omega - z_i)/\varphi(\omega)]_{\omega = z_i}^{(\alpha_i - k - 1)} \neq 0$$

and $\varphi^{(h)}(z_k) = 0, \quad h = 0, 1, \ldots, \alpha_k - 1;\ k = 1, 2, \ldots, s.$

Then the Fourier transform of the set of desired finite controls is equal to

$$\tilde{u}(\omega) = \varphi(\omega) \left\{ \sum_{i=1}^{s} \sum_{k=0}^{\alpha_i - 1} \frac{\beta_i^h}{k!} (\omega - z_i)^{k - \alpha_i} \left[\frac{(\omega - z_i)}{\varphi(\omega)} \right]_{\omega = z_i}^{(\alpha_i - k - 1)} + \gamma(\omega) \right\}, \tag{10}$$

where $\gamma(\omega)$ is the same as in (7). In particular, when $\alpha_i = 1$ for all i, we obtain formula (7).

As the function $\varphi(\omega)$, one may take

$$\varphi(\omega) = \prod_{k=1}^{s} \left(1 - e^{-j\omega T/n}\right)^{\alpha_k}. \tag{11}$$

5. **Finite control in the class of generalized functions.** Above, the solution $u(t)$ was sought in the class $L_2[0, T]$. Such a condition imposes the restriction that as $T \to 0$ the function $u(t)$ tends to a function that is not square-integrable. For example, such a situation occurs in the problem of instantaneously ($T = 0$) bringing system (1) from the initial state $x(0) = x^0$ to the final state $x = 0$ by means of impulsive control actions. Therefore it makes sense to extend the class of admissible finite controls to the class of generalized functions ([1,7]). In this case, as $\tilde{u}(\omega)$ one may take entire functions of finite degree T, which grow as $|\omega| \to \infty$ no faster than $|\omega|^m$. For example,

$$\tilde{u}(\omega) = P_m(\omega), \tag{12}$$

where $P_m(\omega)$ is a polynomial of degree m. Then the inverse generalized Fourier transform ([7]), in symbolic notation, has the form

$$u(t) = P_m(-jd/dt)\delta(t). \tag{13}$$

6. **Example of finite control in the class of generalized functions.** Let the controlled system be described by the equation

$$a_0 x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = u \tag{14}$$

and let the initial conditions be specified:

$$x(0) = x_0, \qquad x'(0) = x_1, \ldots, x^{(n-1)}(0) = x_{n-1}. \tag{15}$$

It is required to find a control $u(t)$ that instantaneously ($T = 0$) brings the system to equilibrium

$$x = x' = x'' = \cdots = x^{(n-1)} = 0. \tag{16}$$

Applying the Fourier transform to equation (14), taking into account the initial conditions (15), solving the resulting equation with respect to $\tilde{x}(\omega)$, and equating the numerator of the expression obtained to zero, we obtain

$$\tilde{u}(\omega) = -\sum_{s=0}^{n-1} \left(\sum_{k=0}^{n-1-s} a_k x_{n-1-s-k} \right) (j\omega)^s. \tag{17}$$

Using formula (13), we obtain that the desired equation has the form

$$u(t) = -\sum_{s=0}^{n-1} \left(\sum_{1}^{n-1-s} a_k x_{n-1-s-k} \right) \delta^{(s)}(t), \tag{18}$$

where $\delta^{(s)}(t)$ denotes the s-th derivative of $\delta(t)$.

7. **Conclusion.** The method proposed in this paper makes it possible to solve boundary-value problems arising in the control of systems. The solutions obtained are not unique. Therefore the functions $\varphi(\omega)$ and $\gamma(\omega)$ appearing in equalities (7) and (10) may be sought from an additional condition under which these functions are such that the desired control $u(t)$ is not only finite, but also optimal in the sense of minimizing some prescribed functional of $u(t)$ and $x(t)$ on $[0, T]$.

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Received
10 V 1967

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Note: Figure translations are in progress. See original paper for figures.