ON THE ABSOLUTE AND UNIFORM CONVERGENCE OF FOURIER SERIES IN EIGENFUNCTIONS OF A SELF-ADJOINT ELLIPTIC OPERATOR OF ORDER \(2m\)

MATHEMATICS

1968

SovietRxiv

Abstract

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UDC 517.9

MATHEMATICS

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ON THE ABSOLUTE AND UNIFORM CONVERGENCE OF FOURIER SERIES IN EIGENFUNCTIONS OF A SELF-ADJOINT ELLIPTIC OPERATOR OF ORDER $2m$

(Presented by Academician A. N. Tikhonov on 26 XII 1967)

In the present paper we study the question of absolute and uniform convergence of Fourier series in the eigenfunctions of a self-adjoint elliptic operator L of order $2m$, given in an arbitrary N-dimensional domain g, i.e., in the eigenfunctions of the equation

$$Lu + \lambda u = \sum_{0 \le |k| \le 2m} a_k(x) D^k u + \lambda u = 0, \tag{1}$$

where $k = (k_1, \dots, k_N)$ is a differentiation index, $|k| = k_1 + \dots + k_N$, $D^k = D^k_x = \partial^k / \partial x_1^{k_1}, \dots, \partial x_N^{k_N}$, with boundary conditions of the form

$$B_j u\big|_\Gamma = \sum_{0 \le |r| \le m_j} b_{rj}(x) D^r u\big|_\Gamma = 0 \tag{2}$$

where $0 \le m_j \le 2m - 1;\ j = 1, \dots, m;\ \Gamma$ is the boundary surface.

The question under study is treated in the works of a number of authors ($^{4-12}$).

In the papers ($^{4-8}$) the question of absolute and uniform convergence of Fourier series in the eigenfunctions of various boundary-value problems for the operator L of second order was studied. The absolute and uniform convergence of the indicated Fourier series was established under the fulfillment of two requirements: 1) $f \in W_2^{[N/2]+1}(g)$, 2) the functions $f, Lf, \dots, L^s f$, where s is some number, satisfy the corresponding boundary condition.

The authors of (^{10}), under the condition that the function f being expanded belongs to the domain of definition of the operator L^τ, where $\tau > N/4m$, prove the uniform and absolute convergence of the Fourier series of the function in the eigenfunctions of the operator L. But in that paper it is not established under what sufficient conditions the function f belongs to the domain of definition of the operator L^τ, where $\tau > N/4m$. In the papers (^{11,12}), for the case when the operator L is an elliptic operator of second order, we found, in a certain sense, definitive conditions on f in the classes W_2^α (with noninteger α) that ensure absolute and uniform convergence of Fourier series in the eigenfunctions of the elliptic operator of second order (i.e., conditions were found for the function f to belong to the domain of definition of L^τ, where $\tau > N/4m$).

In the present note, preserving the same definitive conditions on f as in (^{11,12}), we prove that the Fourier series of the function f in the eigenfunctions of an elliptic operator of arbitrary order $2m$ converges absolutely and uniformly in an arbitrary subdomain g' of the main domain g. In proving the absolute and uniform convergence of the indicated Fourier series for domains g of dimension not divisible by $4m$, substantial difficulties arise in establishing the inequality

$$\sum f_i^2 \lambda_i^{\alpha/m} \leq c\|f\|^2_{W_2^\alpha}.$$

(here λ_i are eigenvalues, and f_i are the Fourier coefficients of the function $f(x)$ with respect to the system of eigenfunctions $\{u_i(x)\}$).

These difficulties are eliminated with the aid of a representation of the kernel of fractional order through the Green’ s function of a parabolic operator, using the properties of this function.

Let the boundary Γ of the domain g and the coefficients $a_k(x)$ and $b_{rj}(x)$ of the operators L and B_j be such that they satisfy the conditions* which ensure the existence of a complete orthonormal system of eigenfunctions $\{u_i(x)\}$, a countable set of nonnegative eigenvalues $\{\lambda_i\}$, as well as the existence of the Green’ s function of the corresponding parabolic operator and the validity for it of the relations

$$0 \leqslant G(t;x,y) \leqslant \begin{cases} c_1 t^{-N/2m} \exp[-c(|x-y|t^{-1/2m})^q], & \text{for } 0 < t \leqslant 1, \\ c_2 \exp[-c_0 t], & \text{for } t > 1 \end{cases} \tag{3}$$

for $x, y \in \bar{g}$, where $q = 2m/(2m-1)$;

$$|D_x^s D_y^{lG}(t;x,y)| \leqslant c(s,l) t^{-(N+s+l)/2m} \exp[-c(|x-y|t^{-1/2m})^q] \tag{4}$$

for $x, y \in g'$, where g' is an arbitrary closed subdomain of the domain g, $0 \leqslant s + l < 4m$;

$$D_x^s D_y^l [G(t;x,y) - Z(t;x,y)] = O(\exp[-ct^{-1/(2m-1)}]) \tag{5}$$

for $x, y \in g'$, where $Z(t;x,y)$ is the fundamental solution and $0 \leqslant s + l \leqslant 1$. We shall call the indicated conditions **conditions A**.

Theorem. *Suppose that conditions A are fulfilled. A function* f*, given in the domain* g*, satisfies the following two requirements: 1)* $f \in W_2^\alpha(g)$*, where* α *is any real number satisfying the inequality* $\alpha > N/2$*; 2) the function* f *is equal to zero in a boundary strip of the domain* g*.*

Then the Fourier series of the function f *converges absolutely and uniformly in an arbitrary closed subdomain* g' *of the domain* g*.*

Remark 1. V. A. Il' in (see ([13])) constructed an example of a function with an absolutely divergent Fourier series, belonging to the class $W_2^{N/2}$. This shows the finality of the conditions of the theorem in the classes W_2^α.

Lemma 1. *If conditions A are fulfilled, then the series of the form*

$$\sum \frac{u_i^2(x)}{\lambda_i^{N/2m+\beta}},$$

where $\beta > 0$*, converges uniformly in the closed domain* g*.*

Lemma 2. *Suppose that conditions A are fulfilled and* f *is an arbitrary function given in the domain* g *and satisfying the following two requirements: 1)* $f \in W_2^{n+\beta}$*, where* $\beta > 0$*; 2) the function* f *is equal to zero in a boundary strip of the domain* g*.*

Then

$$\sum f_i^2 \lambda_i^{(n+\beta)/m} \leqslant c\|f\|^2_{W_2^{n+\beta}}. \tag{6}$$

Let us outline the proof scheme of the indicated assertions. Lemma 1 follows from estimate (3) and from the representation of the Green' s function in the form of a series. To prove Lemma 2, first of all note that from estimate (5) and from the representation of the Green' s function it follows that

$$\frac{\partial G(t;x,y)}{\partial x_i} = -\frac{\partial G(t;x,y)}{\partial y_i} + O\left(t^{-N/2m} \exp\left[-c\left(\frac{|x-y|}{t^{1/2m}}\right)^{2m/(2m-1)}\right]\right). \tag{7}$$

It remains to consider separately four cases: 1) $n/m = 2k$; 2) $n/m = 2k+1$; 3) $n/m = 2k + m_1/m$, where $m_1 < m$; 4) $n/m = 2k+1+m_1/m$, where $m_1 < m$.

[*] Sufficient conditions ensuring the existence of complete orthonormalized functions and a countable set of nonnegative eigenvalues are indicated in paper ([1]), and conditions ensuring the validity of relations (3)–(5) are indicated in papers ([2],[3]).

We shall confine ourselves to indicating the scheme of proof for one of the four cases, for example case 2), since the other cases require only minor changes.

Using the representation of the Green function of the parabolic operator in the form of a series, we obtain

$$\sum f_{hi}^2 e^{-\lambda_i t} = \iint_{gg} G(t;x,y) f_n(y) f_h(x)\, dx\, dy, \tag{8}$$

where $f_h(x)$ is the average of the function $f(x)$ over a certain ball of radius h with center at the point x.

Differentiating (8) $(n+m)/m$ times with respect to t, using the relation $\partial u/\partial t = -Lu$ and applying Green’ s formula, we obtain

$$\sum f_{hi}^2 \lambda_i^{(n+m)/m} e^{-\lambda_i t} =$$

$$= \iint_{gg} G(t;x,y) L^{(n+m)/2m} f_h(x)\, L^{(n+m)/2m} f_h(y)\, dx\, dy.$$

Multiplying both sides of the last equality by t^{α_1}, where $\alpha_1 = (m-\beta)/m$, and integrating with respect to t from 0 to ∞, and then using the equality

$$\iint_{gg} \sum_{\substack{|k|\le m \\ |\nu|\le m}} \sum_{\substack{|l|\le m \\ |s|\le m}} a_{k\nu}(x) a_{ls}(y) D_x^k D_y^{lG}(t;x,y) D_x^\nu p(x) D_y^s p(y)\, dx\, dy =$$

$$= \frac{1}{2} \iint_{gg} \sum_{\substack{|k|\le m \\ |\nu|\le m}} \sum_{\substack{|l|\le m \\ |s|\le m}} a_{k\nu}(x) a_{ls}(y) D_x^k D_y^{lG}(t;x,y) D_x^\nu p(x) D_x^s p(x)\, dx\, dy +$$

$$+ \frac{1}{2} \iint_{gg} \sum_{\substack{|k|\le m \\ |\nu|\le m}} \sum_{\substack{|l|\le m \\ |s|\le m}} a_{k\nu}(x) a_{ls}(y) D_x^k D_y^{lG}(t;x,y) D_y^\nu p(y) D_y^s p(y)\, dx\, dy +$$

$$+ \frac{1}{2} \iint_{gg} \sum_{\substack{|k|\le m \\ |\nu|\le m}} \sum_{\substack{|l|\le m \\ |s|\le m}} a_{k\nu}(x) a_{ls}(y) D_x^k D_y^{lG}(t;x,y) (D_x^\nu p(x) - D_y^\nu p(y))$$

$$\times (D_x^s p(x) - D_y^s p(y))\, dx\, dy,$$

where $p(x) = L^{(n-m)/2m} f(x)$, by relations (5) and (7) and the embedding theorem, we obtain inequality (6). The theorem on absolute and uniform convergence follows directly from the Cauchy-Bunyakovsky inequality and from Lemmas 1 and 2.

In conclusion I express my deep gratitude to V. I. Il' in and A. A. Arsen' ev for valuable advice and comments.

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Received
20 XII 1967

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Note: Figure translations are in progress. See original paper for figures.

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