INTERNAL INVERSE PROBLEMS OF POTENTIAL THEORY

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Abstract

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INTERNAL INVERSE PROBLEMS OF POTENTIAL THEORY

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The problem of determining the shape and density of a body from its internal potential is considered. In the classical theory of the Newtonian potential these problems were studied in connection with the theory of the figure of the Earth and celestial mechanics. Dive ([7]) proved the uniqueness of the solution of the indicated problem in the class of convex bodies of constant density (see ([5])). In the case of ellipsoidal bodies this problem was considered in works ([8], [9]) (see ([5])). In the author' s work ([4]) the question of uniqueness of the solution of the indicated problem was investigated for arbitrary contact bodies of positive density. We note that these problems are equivalent to integral equations of the first kind, methods for solving which are developed in works ([2], [3], [6]) and others. Below we give uniqueness theorems for the solution of the problem for a number of bodies of variable and constant densities.

1°. Consider the uniformly elliptic operator of second order

$$Lu \equiv \sum_{i,k=1}^{n} a_{ik} u_{x_i x_k} + \sum_{k=1}^{n} b_k u_{x_k} + cu, \tag{1}$$

where $a_{ik} = a_{ki} \in C^{(2,\lambda)}(D_0'),\ b_k \in C^{(1,\lambda)}(D_0'),\ C \in C^{(0,\lambda)}(D_0')$ $(0 < \lambda < 1)$ are functions of the point $x = (x_1, \dots, x_n) \in D_0'$; D_0' is a bounded domain of Euclidean space E^n $(n \geqslant 2)$. Suppose, moreover, that the coefficients of operator (1) satisfy the condition

$$c(x) \leqslant 0, \qquad c^*(x) \leqslant 0, \qquad x \in D_0', \tag{2}$$

where c^* is the coefficient of the function v in the expression L^*v (L^* is the operator adjoint to L in the sense of Lagrange).

Let A_α $(\alpha = 1, 2)$ be open sets with boundaries Γ_α; A_α is the union of a finite number of domains $T_{\alpha j}$ with boundaries $S_{\alpha j}$ of class $A^{(1,\lambda)}$, $A_\alpha \subset D_0$, D_0 is an arbitrary domain, $\overline{D}_0 \subset D_0'$. Introduce the generalized potentials of volume masses and of a simple layer (see ([1]))

$$U^\alpha(x) = \int_{A_\alpha} \Omega(x,y)\mu_\alpha(y)\,dy, \qquad V^\alpha(x) = \int_{\Gamma_\alpha} \Omega(x,y)\xi_\alpha(y)\,d_y^s, \tag{3}$$

where $\Omega(x,y)$ is the principal elementary solution of the operator L; $\mu_\alpha(y)$ $[\xi_\alpha(y)]$ are summable bounded functions of the points $y \in A_\alpha[\Gamma_\alpha]$; $\mu_\alpha(y) \neq 0$ $(\xi_\alpha(y) \neq 0)$ almost everywhere for $y \in A_\alpha$ $(y \in \Gamma_\alpha)$.

Introduce the generalized magnetic potential (see ([4]))

$$Z^\alpha(x) = Z(x; A_\alpha, \mu_\alpha; \Gamma_\alpha, \xi_\alpha) = \beta U^\alpha(x) + \gamma V^\alpha(x), \tag{4}$$

where β, γ are real numbers, $\beta^2 + \gamma^2 \neq 0$, and also the functional $(n \geqslant 3)$

$$I(h; A_\alpha \setminus A_0, \mu_\alpha; \Gamma_\alpha, \xi_\alpha) = \beta \int_{A_\alpha \setminus A_0} \mu_\alpha(y)h(y)\,dy + \gamma \int_{\Gamma_\alpha} \xi_\alpha(y)h(y)\,d_y^s, \tag{5}$$

where $A_0 \neq 0$, $\overline{A}_0 = \overline{A}_1 \cap \overline{A}_2$; moreover, for convenience of formulation, in what follows we shall everywhere assume that A_0 is the union of a finite number of simply connected domains.

Lemma 1. In order that the generalized magnetic potentials (4) satisfy the condition

$$Z(x; A_1, \mu_1; \Gamma_1, \xi_1) = Z(x; A_2, \mu_2; \Gamma_2, \xi_2) \quad \text{for } x \in A_0,$$

it is necessary and sufficient that the following relations hold:

1) $\mu_1(y) = \mu_2(y)$ for $y \in A_0$ (if $\beta \neq 0$);

2)
$$I(h; A_1 \setminus A_0, \mu_1; \Gamma_1, \xi_1) = I(h; A_2 \setminus A_0, \mu_2; \Gamma_2, \xi_2), \tag{6}$$

where $h(y)$ is any regular solution of the equation

$$L^*[h(y)] = 0 \quad \text{for } y \in E^n \setminus \overline{D} \ (n \geq 3); \qquad h(y) \to 0 \text{ as } (y) \to \infty, \tag{7}$$

D is an open set consisting of a finite number of domains, $\overline{D} \subset A_0$.

We note that in the case $n = 2$ the conditions (6) and (7) are somewhat different.

2°. Let T_α be bounded domains ($\overline{T}_\alpha \subset D_0,\ \alpha = 1, 2$) with boundaries S_α of class $A^{(1,\lambda)}$; $T_0 = T_1 \cap T_2$, and it is assumed that $T_0 \neq 0$.

Theorem 1. If the metaharmonic potentials (see (4)) $(\chi \geq 0,\ n \geq 2)$ of volume masses $U(x; T_\alpha, \mu_\alpha)$ of convex domains T_α with nonnegative variable densities $\mu_\alpha \in C^1(\overline{T}_\alpha)$ satisfy the condition

$$U(x;T_1,\mu_1)=U(x;T_2,\mu_2)\quad \text{for } x\in T_0,$$

then

$$T_1=T_2;\qquad \mu_1(y)=\mu_2(y),\ y\in T_1.$$

Theorem 2. If $U(x;T_\alpha,\mu_\alpha)$ are harmonic potentials ($\chi=0,\ n\geq 3$) of volume masses (where T_1 is a finite simply connected domain, T_2 an open ball, $\mu_\alpha(y)$ a nonnegative function depending on the distance of the point y to the center of the ball T_2) and satisfy the condition

$$U(x;T_1,\mu_1)=U(x;T_2,\mu_2)\quad \text{for } x\in T_0,\quad T_0\neq 0,$$

then

$$T_1=T_2;\qquad \mu_1(y)=\mu_2(y),\ y\in T_1.$$

We now give a uniqueness theorem for the solution of the problem for harmonic potentials ($\chi=0$) of variable densities, generally speaking not of constant sign.

Denote by (ρ,θ) the spherical coordinates of the point y of the space E^n $(n\geq 3)$; Q_{ρ_0} is the open ball of radius ρ_0 with center at the point O.

Let the functions $\mu_\alpha(y)\in C^1(\overline{D}_0)$ satisfy the condition

$$\mu_\alpha(y)=\delta_\alpha(y)\eta_\alpha(y),\qquad y\in T_\alpha\setminus\overline{Q}_{\rho_0}\quad(\alpha=1,2), \tag{8}$$

where:

a) the function $\delta_\alpha(y)$, generally speaking not of constant sign, satisfies the condition

$$\partial\delta_\alpha/\partial\rho=0; \tag{9}$$

b) the nonnegative function $\eta_\alpha(y)$ satisfies the conditions

$$\frac{\partial}{\partial\rho}\left[\rho^n\eta_\alpha(y)\right]\leq 0\quad \text{for } \rho\geq\rho_0. \tag{10}$$

Theorem 3. If the harmonic potentials of volume masses ($\chi=0,\ n\geq 3$) $U(x;T_\alpha,\mu_\alpha)$ of domains T_α, star-shaped with respect to the common point $O\in T_0$ ($T_0\neq 0,\ \overline{Q}_{\rho_0}\subset T_0$), and of variable densities of class (8)–(10)

satisfy the condition

$$U(x; T_1, \mu_1) = U(x; T_2, \mu_2) \quad \text{for } x \in T_0,$$

then

$$T_1 = T_2; \qquad \mu_1(y) = \mu_2(y), \quad y \in T_1.$$

3°. We shall call the sets A_1 and A_2 *internally contact* if every connected component of the set $D_0 \setminus (\overline{A}_1 \cup \overline{A}_2)$ has a part Γ_* ($\operatorname{mes}\Gamma_* \neq 0$) of the common boundary of the $(n-1)$-dimensional space with one of the components of the set $A_0 = A_1 \cap A_2$ ($A_0 \neq 0$).

Theorem 4. If for internally contact sets A_α ($\alpha = 1, 2$), nonnegative functions $\mu_\alpha \in C^1(\overline{A}_\alpha)$, $\xi \in C^0(\Gamma_\alpha)$, and also nonnegative numbers β and γ ($\beta^2 + \gamma^2 \neq 0$), the equality of the internal generalized magnetic potentials (4) holds, i.e.

$$Z(x; A_1, \mu_1; \Gamma_1, \xi) = Z(x; A_2, \mu_2; \Gamma_2, \xi) \quad \text{for } x \in A_0,$$

then

$$A_1 = A_2$$

and, moreover, $\mu_1(y) = \mu_2(y)$ for $y \in A_1$ (if $\beta \neq 0$).

We note that for generalized magnetic potentials of internally contact sets of variable densities of nonconstant sign, one can state a number of further theorems similar to the theorems for externally contact sets (see (4)).

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Note: Figure translations are in progress. See original paper for figures.