INTEGRAL THEOREMS TAKING LARGE DEVIATIONS INTO ACCOUNT WHEN CRAMÉR’ S CONDITION IS VIOLATED

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Abstract

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INTEGRAL THEOREMS TAKING LARGE DEVIATIONS INTO ACCOUNT WHEN CRAMÉR' S CONDITION IS VIOLATED

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Let ξ_j, $j = 1, 2, \dots$, be independent identically distributed random variables, $\mathbf{M}\xi_j = 0$, $\mathbf{D}\xi_j = 1$. Form the sums $\zeta_n = \xi_1 + \dots + \xi_n$, and let $P_n(x) = \mathbf{P}\{\zeta_n > x\}$. In this note we describe the behavior of the probability $P_n(x)$ for $x > \sqrt{n}$. The order of growth of x from above is not restricted. We shall assume that the ξ_j have a distribution density $p(x)$, and moreover (cf. $(^{2,5})$)*

$$p(x) \sim \exp[-|x|^{1-\varepsilon}], \qquad |x| \to \infty, \qquad 0 < \varepsilon < 1. \tag{*}$$

Let ρ tend to infinity arbitrarily slowly as $n \to \infty$. The following theorem is a certain refinement of a theorem of V. V. Petrov (^{3}).

Theorem 1. Let condition $(*)$ be satisfied. Then, for

$$x < (c_\varepsilon - \delta)(n-1)^{1/(1+\varepsilon)},$$

$$P_n(x) = \left[1 - \Phi\left(\frac{x}{\sqrt{n}}\right)\right] \exp\left\{\frac{x^3}{n^2}\lambda^{[k]}\left(\frac{x}{n}\right)\right\}(1 + o(1)),$$

where $\Phi(x)$ is the distribution function of the normal $(0,1)$ law; $\lambda^{[k]}(z)$ denotes the first k terms of the Cramér series ((^{1}), p. 170), $k > 1/\varepsilon - 1$, and, finally, $c_\varepsilon = (1+\varepsilon)(2\varepsilon)^{-\varepsilon/(1+\varepsilon)}$, while δ is arbitrarily small.

The following theorem is due to S. V. Nagaev (^{4}).

Theorem 2. Under our conditions, for $x > \rho n^{1/2\varepsilon}$,

$$P_n(x) = nP_1(x)(1 + o(1)).$$

The remaining three theorems are stated for the first time.

Theorem 3. Under our conditions, for

$$(c_\varepsilon + \delta)(n-1)^{1/(1+\varepsilon)} < x < n^{1/2\varepsilon}/\rho$$

$$P_n(x) = \frac{n(1+\varepsilon)}{((1-\alpha)x)^\varepsilon}\sqrt{\frac{2\pi n}{1-(n-1)(1-\varepsilon)^\varepsilon/|\alpha|^{1+\varepsilon}}}\,P_{n-1}(\alpha x)P_1((1-\alpha)x)(1+o(1)).$$

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* It should be noted that condition $(*)$ can be weakened considerably. For example, one may assume that:

$$1)\quad p(x) \sim \exp[-x^{1-\varepsilon}], \quad x \to \infty, \qquad \int_{-\infty}^{0} |x|^k p(x)\,dx < \infty, \quad k > \frac{1}{\varepsilon} - 1$$

or

$$2)\quad p(x) \sim x^\alpha \exp[-x^{1-\varepsilon} l(x)], \quad x \to \infty, \qquad \int_{-\infty}^{0} |x|^k p(x)\,dx < \infty, \quad k > \frac{1}{\varepsilon} - 1,$$

where $l(x)$ varies slowly and sufficiently regularly, etc.

Here δ is any sufficiently small positive number, and α is the smaller positive root of the equation

$$\frac{n-1}{x^{1+\varepsilon}} = \frac{\alpha(1-\alpha)^\varepsilon}{1-\alpha}, \qquad \varepsilon > 1/2,$$

and of the equation

$$\frac{n-1}{x^{1+\varepsilon}} = \frac{\alpha(1-\alpha)^\varepsilon}{1-\varepsilon}\left(1 + \sum_{j=3}^{k} j c_j \left(\frac{\alpha x}{n-1}\right)^{j-2}\right), \qquad 0 < \varepsilon \leqslant 1/2,$$

c_j are the coefficients of the Cramér series $\lambda(z)$.

Let us note that $ax < (c_\varepsilon - \delta)(n-1)^{1/(1+\varepsilon)}$, so that the behavior of the probability $P_{n-1}(ax)$ is known to us from Theorem 1.

Theorem 3 takes an especially simple form if $\varepsilon > 1/2$ and $\rho n^{1/(1+\varepsilon)} < x < n^{1/2\varepsilon}/\rho$. In this case

$$P_n(x) = n\sqrt{2\pi n}\left[1 - \Phi\left(\frac{\alpha x}{\sqrt{n-1}}\right)\right]\exp\left[-(1-\alpha)^{1-\varepsilon}x^{1-\varepsilon}\right](1+o(1)).$$

It remains to describe two intermediate cases, when
$(c_\varepsilon - \delta)(n-1)^{1/(1+\varepsilon)} < x < (c_\varepsilon + \delta)(n-1)^{1/(1+\varepsilon)}$ and when
$\dfrac{n^{1/2\varepsilon}}{\rho} < x < \rho n^{1/2}$.

The two following theorems are devoted to them.

Theorem 4. If $(c_\varepsilon - \delta)(n-1)^{1/(1+\varepsilon)} < x < (c_\varepsilon + \delta)(n-1)^{1/(1+\varepsilon)}$,

$$\begin{aligned} P_n(x) = {} & \left[1 - \Phi\left(\frac{x}{\sqrt{n}}\right)\right] \exp\left[\frac{x^3}{n^2}\lambda^{[k]}\left(\frac{x}{n}\right)\right](1+o(1)) + \\ & + n\sqrt{\frac{2\pi n}{1-(n-1)(1-\varepsilon)\varepsilon/x^{1+\varepsilon}}}\left[1 - \Phi\left(\frac{\alpha x}{\sqrt{n-1}}\right)\right] \times \\ & \times \exp\left\{\frac{\alpha^3 x^3}{(n-1)^2}\lambda^{[k]}\left(\frac{\alpha x}{n-1}\right)\right\}\exp\left[-(1-\alpha)^{1-\varepsilon}x^{1-\varepsilon}\right](1+o(1)). \end{aligned}$$

The meaning of the notation is the same as before.

Theorem 5. If $n^{1/2\varepsilon}/\rho < x < \rho n^{1/2}$,

$$\begin{aligned} P_n(x) = {} & \frac{n x^\varepsilon \exp[-x^{1-\varepsilon}]}{\varepsilon e}\exp\left[-\frac{n}{2x^{2\varepsilon}}\right]\int_0^\infty\int_0^\infty \exp[-u-v] \times \\ & \times \left[\Phi\left(\frac{v x^\varepsilon}{\varepsilon\sqrt{n-1}} + \frac{u x^\varepsilon}{\sqrt{n-1}} - \frac{\sqrt{n-1}}{x^\varepsilon}\right) - \Phi\left(\frac{v x^\varepsilon}{\varepsilon\sqrt{n-1}} - \frac{\sqrt{n-1}}{x^\varepsilon}\right)\right] du\, dv\,(1+o(1)) + \\ & + n x^\varepsilon \exp[-x^{1-\varepsilon}]\int_{-\infty}^0 \left[1 - \Phi\left(\frac{u x^\varepsilon}{\sqrt{n-1}}\right)\right]\exp[(1-\varepsilon)u]\, du\,(1+o(1)). \end{aligned}$$

Thus, Theorems 1-5 completely describe the behavior of the probability $P_n(x)$. Let us examine what gives rise to the large deviations when condition (*) is satisfied. Let x_m have the property that

$$n\int_{x_m}^\infty u^m \exp[-u^{1-\varepsilon}]\, du = o(1).$$

1. In the case of normal deviations $\sqrt{n} < x < C\sqrt{n}$ (C is an arbitrarily large constant), the event $\{\xi_n > x\}$ is formed mainly at the expense of realizations $\xi_1, \dots, \xi_n$ in which $|\xi_i| < x_0$, $i = 1, \dots, n$.
2. If $C\sqrt{n} < x < (c_\varepsilon - \delta)(n-1)^{1/(1+\varepsilon)}$, then it is necessary to distinguish two cases: $\varepsilon > 1/2$ and $0 < \varepsilon \le 1/2$. In the first case the principal contribution to the event $\{\zeta_n > x\}$ is made by realizations $\xi_1, \dots, \xi_n$ for

which $|\xi_i| < x_2,\ i = 1, \dots, n$. In the second case $|\xi_i| < x_m,\ i = 1, \dots, n$, where m is determined from the condition

$$n \left(\frac{x}{n}\right)^m = o(1).$$

We note that, in any case, $m \leq [1/\varepsilon] + 1$.

3. If $(c_\varepsilon - \delta)(n-1)^{1/(1+\varepsilon)} < x < n^{1/2\varepsilon}/\rho$ and $\varepsilon > 1/2$, then the event $\{\zeta_n > x\}$, roughly speaking, is arranged as follows: $n-1$ random variables lie in the interval $(-x_2, x_2)$, and one lies in $((1-\alpha)x - \rho\sqrt{n}, (1-\alpha)x + \rho\sqrt{n})$. If, however, $(c_\varepsilon - \delta)(n-1)^{1/(1+\varepsilon)} < x < n^{1/2\varepsilon}/\rho$ and $0 < \varepsilon \leq 1/2$, then $n-1$ random variables lie in the interval $(-x_m, x_m)$, where m is determined by the requirement $nx^{-m\varepsilon} = o(1)$, and one falls into the interval $(x, x + \rho\sqrt{n})$.
4. If $x > \rho n^{1/2\varepsilon}$, then the event that interests us consists of realizations in which $n-1$ random variables lie in the interval $(-x_0, x_0)$, and one lies in $(x, x + \rho\sqrt{n})$.
5. If

$$\frac{n^{1/2\varepsilon}}{\rho} < x < \rho n^{1/2\varepsilon},$$

then $n-1$ random variables lie in the interval $(-x_2, x_2)$, and one lies either in the interval $(-\rho\sqrt{n} + x, x)$ (which gives rise to the first term in Theorem 5), or in the interval $(x, x + \rho\sqrt{n})$ (which gives rise to the second term).

6. If, however, $(c_\varepsilon - \delta)(n-1)^{1/(1+\varepsilon)} < x < (c_\varepsilon + \delta)(n-1)^{1/(1+\varepsilon)}$, then large deviations are formed partly by realizations characteristic of case 2, and partly by realizations characteristic of case 3.

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