STUDY OF APPROXIMATELY FINITE VON NEUMANN ALGEBRAS WITH FINITE TRACE

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Abstract

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STUDY OF APPROXIMATELY FINITE VON NEUMANN ALGEBRAS WITH FINITE TRACE

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1. Among factors of type II_1, the simplest properties are possessed by approximately finite factors. Recall that an approximately finite factor of type II_1 is a factor M with an exact, normal, and finite trace; moreover, there exists an increasing sequence of factors $\{M_n\}_1^\infty$ of type I_n $(n < \infty)$, generating M.

Let us recall the results of Murray—von Neumann on approximately finite factors ([2]).

Let A be a commutative von Neumann algebra with exact normal and finite trace Tr_A, and let G be a discrete group of $*$-automorphisms of A preserving the trace. If G is ergodic and freely acting, then the crossed product $G \times A$ is a factor of type II_1. From the assumption that G is an increasing sequence of finite groups, it follows that $G \times A$ is an approximately finite factor.

Murray and von Neumann formulated the theorem:

Let G be a discrete commutative group of automorphisms of A, preserving the trace; then $G \times A$ is also an approximately finite factor.

This theorem was proved in full only recently by H. Dye ([3,4]), who obtained profound results in the theory of measure-preserving transformations.

In the present article we have attempted to generalize these facts under the assumption that the commutative $*$-ring A is replaced by an approximately finite weakly closed $*$-ring M with finite trace. The results found are used to study factor representations of type II_1 of discrete nilpotent groups.

We note that the theory of crossed products as applied to factors of type II_1 was developed in ([1]). We have tried to preserve the notation used in that work.

2. We first study cyclic groups of automorphisms of approximately finite factors.

Theorem 1. Let M be an approximately finite factor of type II_1, whose elements are operators in a separable Hilbert space H. Let G be a cyclic group of order n $(n = 2, 3, ...)$ of outer automorphisms of M. Then the crossed product $\mathfrak{M} = G \times M$ is an approximately finite factor of type II_1.

The proof of the theorem is based on the following lemmas.

Lemma 1. Let M be a factor of type II_1, and let G be a cyclic group of outer automorphisms of the factor M of order n $(n = 2, 3, ...)$. Denote by M_0 the set of fixed elements of M relative to the group G. Then M_0 is a factor of type II_1.

If M_0 is an approximately finite factor of type II_1, then $\mathfrak{M} = G \times M$ is also an approximately finite factor of type II_1. The converse is true.

Lemma 2. Let M be a Neumann algebra with a faithful, normal and finite trace Tr_M, whose elements are operators in the Hilbert space H. Let ε be a primitive root of unity of the n-th degree, where n is a natural number $(n = 2, 3, ...)$.

In order that the ring M be an approximately finite factor of type II_1, it is necessary and sufficient that M be the minimal weakly closed ring generated by a family of operators $\{w_i\}_1^\infty$ in H possessing the following properties:

1) $w_i^n = 1,\ w_i^k \neq 1\ (0 < k < n),\ i = 1, 2, ...;$
2) $w_{i-1} w_i = \varepsilon w_{i w_{i-1}},\ w_{iw} j = w_{jw} i \quad (j \neq i-1, i+1).$

Lemma 3. Let M be a factor of type II_1, whose elements are operators in the Hilbert space H. Let N_1 and N_2 be subfactors of M of type I_n. Then there exists an automorphism φ of the factor M which maps N_1 isomorphically onto N_2.

We outline the proof of Theorem 1. Denote by g a generator of G, and by U_g the corresponding unitary operator from $\mathfrak{M} = G \times M$. Consider the automorphism h of the factor $\mathfrak{M}$, which we define as follows:

$$(U_g)^h = \varepsilon U_g; \qquad (1 \otimes m)^h = 1 \otimes m, \qquad m \in M,$$

where ε is a primitive root of unity of the n-th degree. It is easy to prove that h is an outer automorphism of $\mathfrak{M}$ and $h^n = 1$. Denote by G_h the group generated by h, and construct $G_h \times \mathfrak{M}$. Since the set of elements of $\mathfrak{M}$ fixed with respect to h is the approximately finite factor M, it follows, according to Lemma 1, that $G_h \times \mathfrak{M}$ is also an approximately finite factor of type II_1. Consequently, in $G_h \times \mathfrak{M}$ there exists a family of operators $\{w_i\}_1^\infty$ possessing properties 1) and 2) of Lemma 2 and generating $G_h \times \mathfrak{M}$.

Let h correspond to the unitary operator $U_h \in G_h \times \mathfrak{M}$. Consider now the subring of $G_h \times \mathfrak{M}$ generated by the operators U_g, U_h. A simple verification shows that this ring is a factor of type I_n. The ring generated by the operators w_1 and w_2 is also a factor of type I_n. It follows from Lemma 3 that there exists an automorphism φ of the factor $G_h \times \mathfrak{M}$ for which

$$\varphi(w_1) = U_h, \qquad \varphi(w_2) = U_g.$$

Put

$$\varphi(w_k) = U_k \qquad (k = 3, 4, \dots).$$

Then the family of operators U_g, U_k $(k = 3, 4, \dots)$ generates the factor $\mathfrak{M}$ and possesses properties 1) and 2) of Lemma 2. From this lemma we conclude that $\mathfrak{M}$ is an approximately finite factor, as was required to be established.

As a corollary we obtain the theorem:

Theorem 2. Let M be an approximately finite factor of type II_1. Let G be a solvable finite group of outer automorphisms of M. Then $G \times M$ is an approximately finite factor of type II_1.

3. Definition. A Neumann algebra M with a faithful normal and finite trace Tr_M is called **approximately finite** if, for an arbitrary number $\varepsilon > 0$ and for every finite subset of operators $x_1, \dots, x_s$ from M, there exists a subalgebra N of type I, whose center Z_N contains the center Z_M of the algebra M, and which contains operators $x'_1, \dots, x'_s$ such that

$$[[x_i - x'_i]] = \mathrm{Tr}_M((x_i - x'_i)(x_i - x'_i)^*)^{1/2} < \varepsilon \qquad (i = 1, \dots, s).$$

In what follows we shall assume that the center Z_M contains no minimal projections.

Theorem 3. Let $\mathfrak{M}$ be a factor of type II_1, whose elements are operators in a separable Hilbert space H. Let M be an approximately finite subalgebra of $\mathfrak{M}$, whose center Z_M is distinguished from $\{\lambda I\}$, where I is the identity operator in H.

Denote by $N(M)$ the normalizer of M in $\mathfrak{M}$, i.e., the set of unitary operators $U \in \mathfrak{M}$ such that $\varphi_U(M) = U^{-1}MU = M$. We note that $\varphi_U(Z_M) = Z_M$ for $U \in N(M)$. Denote the corresponding group of automorphisms of the center Z_M by G.

Suppose that the weak closure of $N(M)$ coincides with $\mathfrak{M}$, and that to each $U \in N(M)$, $U \notin M$, there corresponds a nonidentity automorphism φ_U of the center Z_M. If G is approximately finite in I', the group of automorphisms of Z_M, then $\mathfrak{M}$ is an approximately finite factor.

G. Dye proved this theorem under the assumption that M is a commutative subalgebra of $\mathfrak{M}$ ([4]).

Corollary. Let M be an approximately finite Neumann algebra with finite trace. Suppose that the center Z_M is different from $\{\lambda I\}$. Denote by G the commutative group of outer automorphisms of M such that to each $g \in G$ there corresponds a freely acting automorphism of the center Z_M.

Then the crossed product $\mathfrak{M} = G \times M$ is an approximately finite Neumann algebra.

If G is an ergodic group of automorphisms of the center Z_M, then $\mathfrak{M} = G \times M$ is an approximately finite factor of type II_1.

Theorem 3 has an interesting application in the theory of representations of discrete groups.

Theorem 4. Let a discrete nilpotent group G have a factor representation $g \to U_g$ of type II_1 in the Hilbert space H. Then the factor M, generated by the operators U_g, $g \in G$, is approximately finite.

The proof of the theorem relies essentially on the following result.

Lemma 4. Let a discrete nilpotent group G have a commutative normal divisor K, which is a maximal commutative subgroup of G, i.e., the centralizer $C(K)$ of the group K in G coincides with K. If $g \to U_g$ is an exact factor representation of type II_1 of the group G in the Hilbert space H, then the factor M, generated by the operators U_g, $g \in G$, is approximately finite.

We outline the proof of the lemma. Recall that a representation is called exact if operators of the form λI correspond only to elements of the center Z_0 of the group G. From the assumption of nilpotency of the group G it follows that there exists a sequence of subgroups:

$$G = C(Z_0) \supset C(Z_1) \supset \dots \supset C(Z_t),$$

where Z_k is a subgroup of K and $Z_0 \subset Z_1 \subset \dots \subset Z_t = K$, while $C(Z_k)$ is the centralizer of Z_k in $C(Z_{k-1})$, and the commutator group $(C(Z_{i-1}), Z_i) \subset Z_{i-1}$ $(i = 1, \dots, t-1)$. From the assumption on the group K it follows that $C(Z_t) = K$. Further, each group $C(Z_i)$ is a normal divisor of $C(Z_{i-1})$. Since $(C(Z_{i-1}), Z_i) \subset Z_{i-1}$, the group $C(Z_{i-1})/C(Z_i)$ induces a commutative group of automorphisms of the center Z_i of the group $C(Z_i)$. Indeed, if $k \in Z_i$, $\tilde{x}, \tilde{y} \in C(Z_{i-1})/C(Z_i)$, then

$$x^{-1}kx = k_{xk}, \qquad y^{-1}ky = k_{yk} \qquad (x \in \tilde{x},\ y \in \tilde{y}),$$

where $k_x, k_y \in Z_{i-1}$, and therefore

$$y^{-1}(x^{-1}kx)y = y^{-1}k_{xky} = k_{xk_{yk}} = k_{yk_{xk}} = x^{-1}(y^{-1}ky)x.$$

Denote by M_i $(i = 0, 1, \dots, t)$ the weakly closed ring generated by the operators U_g, where $g \in C(Z_i)$. Then M_t is a commutative subring of M; consequently, M_t is approximately finite. Suppose that M_i is approximately finite, and prove the approximate finiteness of M_{i-1}, where $M_i \subset M_{i-1}$. We note that the center of M_i is generated by the operators U_g, where $g \in Z_{i-1}$, and, as we have seen, the operators U_g, where $g \in C(Z_{i-1})$, induce a commutative group $C(Z_{i-1})/C(Z_i)$ of automorphisms of the cent-

From Theorem 3 we conclude that M_{i-1} is approximately finite. This completes the proof of the lemma.

The proof of Theorem 4 requires only minor complications.

Example. Let G be the group of triangular matrices of order n $(n = 2, 3, ...)$ with ones on the main diagonal over the field of rational numbers or over the ring of integers. It is not difficult to show that the regular representation of this group decomposes into a direct integral of factor representations of type II_1. It follows from Theorem 4 that these factors are approximately finite.

Theorem 5. *Let G be a discrete group. Then its regular representation generates a Neumann algebra with a finite, faithful, and normal trace.*

This result is well known for discrete commutative groups (Pontryagin's theorem) and for finite groups.

In particular, the regular representation of a discrete nilpotent group generates an approximately finite Neumann algebra with a finite trace.

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Note: Figure translations are in progress. See original paper for figures.