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Abstract

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MATHEMATICS

K. ESHMURADOV

CONDITIONS FOR STRUCTURAL STABILITY AND CLASSIFICATION OF CELLS OF A THREE-DIMENSIONAL DYNAMICAL SYSTEM WITHOUT CLOSED TRAJECTORIES

(Presented by Academician A. Yu. Ishlinskii, 1 III 1968)

In the works of A. A. Andronov, L. S. Pontryagin, E. A. Leontovich, and A. G. Maier ([1]), conditions for structural stability were established and a classification was given of the cells of a two-dimensional dynamical system. In the present paper we formulate the results of solving these same questions for a dynamical system of third order, described in a certain domain G (bounded by a certain smooth contact-free surface S) by the equations

$$dx/dt = P(x, y, z), \qquad dy/dt = Q(x, y, z), \qquad dz/dt = R(x, y, z) \tag{1}$$

under the essential restriction that closed trajectories are absent.

Necessary and sufficient conditions for structural stability with respect to small perturbations, together with the first derivatives, of the right-hand sides of the dynamical system (1) can be formulated in the form of the following theorem:

Theorem 1. *The dynamical system (1) without closed trajectories is structurally stable in the domain G if and only if in this domain it:*

1) *has only such equilibrium states for which the corresponding characteristic equations have no roots with zero real part, and the number of such equilibrium states is finite;*
2) *has no semitrajectories stable in the sense of Poisson other than equilibrium states;*
3) *has only a finite number of trajectories going from a saddle or saddle-focus into another saddle or saddle-focus, along which the separatrix surfaces of the corresponding saddles or saddle-foci intersect without tangency;*
4) *has no trajectories going from a saddle to a saddle, except for the trajectories listed in the preceding item;*
5) *has no limiting set of semitrajectories other than an equilibrium state.**

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From the structural-stability conditions 1–5 it follows that a structurally stable dynamical system of third order without closed trajectories can have phase trajectories only of the following types**:

I. Equilibrium state:

1) a stable or unstable node (focus);

2) a stable or unstable saddle (saddle-focus)***.

II. From an unstable (stable) saddle or saddle-focus:

3) tending as $t \to -\infty$ ($t \to +\infty$) to an unstable (stable) node or focus;

4) leaving the domain G as $t \to -\infty$.

* Conditions 1–5 agree with Smale' s well-known hypothesis ([2]). The proofs of items 1 and 2 are based on the results of ([3]) and on the lemma on closures of ([4]). In the proof of items 3 and 4, considerations stated in ([5]) are used.

** Without loss of generality it is assumed that trajectories intersecting the contact-free boundary surface S, with increasing time t, enter the interior of the domain G.

*** A saddle or saddle-focus will be called stable (unstable) if the corresponding characteristic equation has two roots with negative (positive) real parts.

III. The separatrix of a stable (unstable) saddle or saddle-focus:

5) tending as $t \to -\infty$ ($t \to +\infty$) to an unstable (stable) node or focus;

6) leaving the domain G as $t \to -\infty$;

7) tending as $t \to \infty$ ($t \to +\infty$) to an unstable (stable) saddle or saddle-focus whose separatrix surfaces intersect without tangency.

IV. A trajectory tending as $t \to +\infty$ to a stable node or focus;

8) tending as $t \to -\infty$ to an unstable node or focus;

9) leaving the domain G as $t \to -\infty$.

With respect to the singular and ordinary trajectories defined in accordance with paper ([1]), the following theorem holds.

Theorem 2. *Trajectories of types I–III are singular, and trajectories of type IV are ordinary.*

The set of points belonging to ordinary trajectories is open and may decompose into a finite or infinite number of connected domains, which are called cells ([1]). The number and the basic characteristics of the cells are determined by the following theorem.

Theorem 3. *If a dynamical system without closed trajectories (1) satisfies in the domain G conditions 1–5 of Theorem 1, then:*

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1) *the set of singular trajectories divides the domain G into a finite number of cells*;

2) *each interior cell consists of entire ordinary trajectories and has in its boundary exactly one sink and one source;*

3) *each cell adjacent to the surface S consists of ω-orbitally stable positive semitrajectories and has in its boundary exactly one sink and a connected domain of the surface S.*

The behavior of singular trajectories on the boundary of a cell is described by

Theorem 4. *Every nonsingular separatrix of an unstable (stable) saddle that is part of the boundary of an interior cell, as $t \to +\infty$ ($t \to -\infty$) tends to the sink (source) that is part of the boundary of the given cell. Every whisker of a stable (unstable) saddle that is part of the boundary of an interior cell, as $t \to +\infty$ ($t \to -\infty$) tends to the sink (source) that is part of the boundary of the given cell. Every nonsingular separatrix of an unstable (stable) saddle that is part of the boundary of a cell adjacent to the surface S, as $t \to +\infty$ ($t \to -\infty$) tends to the sink that is part of the boundary of the given cell (leaves the domain G). Every whisker of a stable (unstable) saddle that is part of the boundary of a cell adjacent to the surface S, as $t \to +\infty$ ($t \to -\infty$) tends to the sink that is part of the boundary of the given cell (leaves the domain G).*

Table 1

	$k=0$	$k=1$	$k=2$					
	$n=0$	$n \geq 1$	$n=0;\ 1$	$n \geq 2$	$n=0;\ 1$	$n=2$	$n=3;\ 4$	$n \geq 5$
α	0	n	0	$n-2$	0	1	$2n-4$	$3n-8$
β	1	$n+1$	0	$n-1$	0	1	$4n-8$	$4n-8$

Special separatrices are separatrices going from a saddle to a saddle; all the remaining separatrices are called nonsingular.

The type of a cell is characterized to a considerable extent by the numbers n and k of saddles and special separatrices that are part of the boundary of the cell. In Table 1, for a number of the first values of k and arbitrary n, the numbers α and β of topological–

* In the presence, in a third-order dynamical system, of closed trajectories, as shown by the example constructed in paper ([6]), the number of cells can be infinite.

sketches of different types of internal cells and cells adjacent to the surface S.

Below is a clarification of the composition and character of the cell boundaries.

For any $n \geq 1$ and $k = 0$, the boundary of an internal cell includes n nodes and all separatrices of each saddle that is part of the boundary of the given cell;

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and the boundary of a cell adjacent to the surface S, in addition, includes an n-connected region of the surface S. For $n = k = 0$, the boundary of a cell adjacent to the surface S consists of a sink and the surface S.

(Figure: Fig. 1)

Fig. 1

(Figure: Fig. 2)

Fig. 2

For any $n \geq 3$ (for cells adjacent to the surface S, also for $n = 2$) and $k = 1$, the boundary of an internal cell includes one special separatrix, $(n+2)$ nodes, and all separatrices of each saddle that is part of the boundary of the given cell; and the boundary of a cell adjacent to the surface S, in addition, includes an $(n-1)$-connected region of the surface S.

For $k = 2$, the boundary of a cell includes two special separatrices γ_1 and γ_2. If γ_1 and γ_2 go from one and the same saddle to one and the same saddle, then we shall call the given cell a cell of the first kind; if from one and the same saddle to different saddles—a cell of the second kind; and, finally, if from different saddles to different saddles—a cell of the third kind.

For any $n \geq 2$ and $k = 2$, the boundary of an interior cell of the first kind includes n whiskers, all separatrices of $(n-2)$ saddles, and part of the nonspecial separatrices of two saddles that are contained in the boundary of the given cell; and the boundary adjoining the surface S of a cell of the first kind, in addition, includes an $(n-1)$-connected region of the surface S.

For any $n \geq 3$ and $k = 2$, the boundary of an interior cell of the second kind includes $(n+1)$ whiskers, all separatrices of $(n-1)$ saddles, and part of the nonspecial separatrices of one saddle; and the boundary adjoining the surface S of a cell of the second kind, in addition, includes an $(n-2)$-connected region of the surface S. For $n = k = 2$ there are no cells of the second kind.

For $n \geq 5$ (for cells adjoining the surface S, also for $n = 4$) and $k = 2$, the boundary of an interior cell of the third kind includes $(n+4)$ whiskers and all separatrices of each saddle contained in the boundary of the given cell; and the boundary adjoining the surface S of a cell of the third kind, in addition, includes an $(n-2)$-connected region of the surface S. For $n = 2, 3$ and $k = 2$ there are no cells of the third kind, and for $n = 4$ and $k = 2$ there are no interior cells of the third kind.

The cells described above are rough.

In Figs. 1 and 2b there are schematically shown an interior cell with $k = 0$, a cell with $k = 0$ adjoining the surface S, and an interior cell with $k = 1$. In Fig. 2a there is shown an element whose insertion into the interior of a cell with $k = 0$ gives a cell with $k = 1$. In these figures the following notation is introduced: O^+—sink; O^-—source; Π^0—region of the surface S contained in the

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boundary of the cell; C_i^-—unstable saddle; C_i^+—stable saddle; m—the number of unstable saddles and $(n-m)$—the number of stable saddles contained in the boundary of the given cell; γ—a trajectory going from saddle to saddle (a special separatrix).

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Scientific Research Institute
of Applied Mathematics and Cybernetics
at Gorky State University
named after N. I. Lobachevsky

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