ON \(k\)-STABILITY OF HOMEOMORPHISMS AND THE UNION OF CELLS

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Abstract

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ON k-STABILITY OF HOMEOMORPHISMS AND THE UNION OF CELLS

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In the present note we extend certain results obtained on the basis of the technique of local correction of homeomorphisms (see ([1])) to dimensions where they were not yet known. We propose here the indicated known technique and point out one new device which makes it possible to apply it in these dimensions.

The first result is a reduction in the question of stability of homeomorphisms of the sphere (see ([1])). In ([1]) the concept of a k-stable homeomorphism was introduced, and in ([2]) it was proved that every homeomorphism of the n-dimensional sphere is $(n-3)$-stable. We shall show that $(n-1)$-stability follows from $(n-2)$-stability and, consequently, as is well known (see ([6])), stability of homeomorphisms of degree 1. Thus the whole problem is reduced to proving $(n-2)$-stability. As in ([2]), we use here the theorem on the union of cells ([4]), namely of cells of codimension 1. This theorem has hitherto been proved for S^n when $n \neq 4$. We give here its proof also for $n = 4$, but in a weaker formulation (such as in ([5])), which is insufficient for use in Theorem 1. In view of this, Theorem 1 is proved here only for $n \geq 5$. However, for other applications, in particular for the proof of the theorem on the absence of isolated singularities in codimension 1, the assertion of Theorem 2 is sufficient.

1. Notation and formulations of the results. Let $p(t) = e^{2\pi i t}$ be the standard covering $p : R \to C$ of the circle C of the complex plane by the real line. Put $a_t = e^{2\pi i t}$. Represent the sphere S^n as the join $S^{n-2} * C$ and put $B_t = S^{n-2} * a_t$, $B_0 = B$, $a_0 = 0$–the center of B, $Q(t_1, t_2) = S^{n-2} * [a_{t_1}, a_{t_2}]$. Let $P : B \times R \to S^n$ be the mapping which sends $r_x \times t$, where r_x is the radius of B drawn to the point $x \in S^{n-2}$, linearly onto the segment joining x and a_t. On $\operatorname{Int} B \times R$ this mapping is the universal (cyclic) covering of the complement of S^{n-2} in S^n. Note also that $P(x \times R) = x$ for $x \in S^{n-2}$.

Theorem 1. *If $h : S^n \to S^n$, $n \geq 5$, is a homeomorphism identical on S^{n-2}, then there exists a stable homeomorphism $\tilde{h} : S^n \to S^n$ such that $\tilde{h}h$ is identical on B.*

Corollary. *If a homeomorphism of the sphere S^n onto itself is $(n-2)$-stable, then it is $(n-1)$-stable, and if it preserves orientation, then it is stable.*

For the formulation of the second result, denote by B_+ and B_- two semidisks which in the standard way give B in their union.

Theorem 2. *Let $q : B \to S^n$ be an embedding such that $q|B_+$ and $q|B_-$ are locally flat. Then q is locally flat.*

We note that for $n \geq 5$ and $n \leq 3$ we have a stronger result, since there it is enough that one of the disks be embedded locally flatly only at interior points (see ([4],[7],[8])).

Corollary. *The local flatness of an embedding of a three-dimensional manifold in S^4 cannot be violated at isolated points (see ([4])).*

2. Proof of Theorem 1.

Preliminary remark. If $hB \subset Q(-\frac{1}{4}, \frac{1}{4})$, then, according to the theorem on the union of cells, $hB \cup B_{1/2}$ would be a ···

a locally flat sphere, and an embedding $q : B \cup B_{1/2} \to S^n$, equal to h on B and to 1 on $B_{1/2}$, would be stably equivalent to the identity (see Corollary 2 in ([1]) and ([2])). Hence, as in ([2]), it follows that already the embedding $h/B \cup B_{1/2}$ is stably equivalent to the identity and, in particular, so is h/B. Thus, if one could construct a stable homeomorphism $\bar{h} : S^n \to S^n$ such that $\bar{h}hB \subset Q(-1/4, 1/4)$, the theorem would be proved. We shall show, however, assuming that $h0 = 0$, that there is an embedding $q : B \to S^n$ such that: a) $q/S^{n-2} = 1$; b) $qB \subset Q(-1/4, 1/4)$; c) $q = h$ on some neighborhood O in B. Then, according to what was said above, the embedding $q : B \cup B_{1/2} \to S^n$, equal to q on B and to the identity on $B_{1/2}$, is stable. Since $q = q = h$ on some neighborhood O in B, q belongs to the same stable class as $h/B \cup B_{1/2}$, and hence $h/B \cup B_{1/2}$ is stable.

Proof of the theorem. In view of the remark just made, it suffices for us to construct an embedding q with the three properties indicated above. Construct a map $r : \operatorname{Int} B \to (\operatorname{Int} B) \times R$, covering h, such that $r0 = 0 \times 0$. Clearly, r is an embedding. Observe that when a point x moves in $\operatorname{Int} B$, approaching a point $x_0 \in S^{n-2}$, the point rx approaches the line $x_0 \times R$, although, possibly, it does not tend to any point. It is easy to construct a continuous function $\varphi(x) > 0$, $x \in \operatorname{Int} B$, such that $r(\operatorname{Int} B) \cap (x \times R)$ lies in the interval $x \times [-\varphi(x), \varphi(x)]$. Construct on each line $x \times R$ a homeomorphism H_x which carries the points $(x \times \varphi(x))$ and $(x \times -\varphi(x))$, respectively, to the points $(x \times 1/4)$ and $(x \times -1/4)$, is fixed on the segment $x \times [-1/8, 1/8]$, and is linear on the complementary intervals (on the infinite ones, a shift without stretching). It is clear that H_x depends continuously on the point x, in view of the continuity of $\varphi(x)$, and hence a homeomorphism $H : (\operatorname{Int} B) \times R \to (\operatorname{Int} B) \times R$ is defined with the following properties: 1) $Hr(\operatorname{Int} B) \subset (\operatorname{Int} B) \times [-1/4, 1/4]$; 2) H is the identity on $(\operatorname{Int} B) \times [-1/8, 1/8]$; 3) H carries each line $x \times R$ onto itself.

Observe that by 2) and by the fact that $r0 = 0 \times 0$, $Hr = r$ on some neighborhood

O in B (whose image lies in $\operatorname{Int} B \times [-1/8, 1/8]$). By 1), $Hr(\operatorname{Int} B)$ belongs entirely to the interior of one sheet of the covering P, namely, $(\operatorname{Int} B) \times [-1/2, 1/2]$, and therefore $q = PHr$ is an embedding. Further, by 3), this embedding extends uniquely to the identity mapping on S^{n-2}. Finally, by 1), $qB \subset S^{n-2} * [-1/4, 1/4]$, and since Hr coincides with r on some neighborhood O in B, q coincides with $h/B = Pr$ on the same neighborhood O in B.

Thus q with the required properties has been constructed and, by the remark made above, the theorem is proved.

3. Proof of Theorem 2

Preliminary remark. We shall first prove a somewhat weaker form of the theorem, namely, we shall assume that the cells have their common boundary as their intersection. In doing so we shall prove, in essence, only the applicability to our case of the proof given in ([4]). Then we shall show how to reduce to this case the case when the cells intersect in a common cell on the boundary.

Proof of the theorem. Thus, suppose first that an embedding $q : S^{n-1} \to S^n$ is given, where $S^{n-1} = B \cup B_{1/2}$, and $q/B_{1/2}$ and q/B are locally flat. We may assume that q is the identity on $B_{1/2}$ and that $B \subset Q(-1/4, 1/4)$ (see ([4,5])). Moreover, according to Lemma 0 of ([3]), there is a homeomorphism $\tilde{q} : S^n \to S^n$ which agrees with q on B, and it may be assumed that already $\tilde{q}Q(-1/4, 1/4) \subset Q(-1/4, 1/4)$.

Turning to the proof of the theorem on the union of cells, we see that, in order to be able to carry this proof over to our case, it suffices for us to be able to prove the following proposition. (Here O_α denotes the α-neighborhood of S^{n-2} in S^n for any $\alpha > 0$.)

Lemma (cf. the lemma in ([4])). *For every $\varepsilon > 0$ one can find such*

$\delta > 0$ and such an ε-homeomorphism $h : S^n \to S^n$, identical outside O_ε, on $Q(3/8,\ 5/8)$ and on $\tilde{q}Q(1/8,\ -1/8)$, that $h\tilde{q}Q(-1/4,\ 1/4) \subset Q(-1/4,\ 1/4) \cap Q_\delta$.

Proof of the lemma. It is easy to construct a mapping

$$r : (\operatorname{Int} B) \times [0, 1] \to B \times R,$$

having the property that $Pr_t = \tilde{q}P_t$, where r_t and P_t are the restrictions of r and P to $(\operatorname{Int} B) \times t,\ t \in [0, 1]$. We set $(\operatorname{Int} B) \times 0 = \operatorname{Int} B$ and assume that $\tilde{q}0 = 0,\ r_0(0) = 0 \times 0$. Then

$$r((\operatorname{Int} B) \times [0, 1/4]) \subset Q(-1/4,\ 1/4)$$

and

$$r((\operatorname{Int} B) \times [3/4, 1]) \subset Q(3/4,\ 5/4).$$

Let $H : B \times R \to B \times R$ be a homeomorphism which carries each line $x \times R$ onto itself, with $H = 1$ on $B \times [-1/4,\ 1/4]$ and

$$H(B \times [3/4,\ 5/4]) = B \times [5/16,\ 3/8].$$

Now let

$$u : B \times [0,1] \to B \times [0,1]$$

be a homeomorphism which, on each segment $x \times [0,1]$, carries the point $x \times 1/4$ to the point $x \times 3/4$, is fixed on $x \times [0,1/8]$ and at the point $x \times 1$, and is linear on adjacent intervals. Then the homeomorphism $v = (Hr)u(Hr)^{-1}$ carries $Hr((\operatorname{Int} B) \times [0,1])$ onto itself and can be extended identically over the remaining part $(\operatorname{Int} B) \times R$. Note that

$$vr((\operatorname{Int} B) \times [1/4,\ -1/4])$$

contains the region between $r((\operatorname{Int} B) \times 0)$ and $(\operatorname{Int} B) \times 1/4$. By means of a symmetric construction we obtain that v can be constructed so that

$$vr((\operatorname{Int} B) \times [-1/4,\ 1/4]) \supset (\operatorname{Int} B) \times [-1/4,\ 1/4],$$

and, moreover, so that $v = 1$ on $(\operatorname{Int} B) \times [-1/8,\ 1/8]$ and also on $(\operatorname{Int} B) \times (3/8, \infty)$ and on $(\operatorname{Int} B) \times (-\infty, -3/8)$. If we now consider the mapping PvP^{-1}, then it defines a homeomorphism on S^n (since v is the identity outside one sheet $B \times [-1/2,\ 1/2]$ of the covering) which satisfies all the requirements of the lemma except those connected with the prescribed $\varepsilon > 0$. In order to satisfy also this last condition, we shall now replace the homeomorphism u in the expression for v by a homeomorphism $\bar{u}$, which coincides with u on $((\operatorname{Int} B) \setminus B') \times R$ and coincides with the identity on $B'' \times R$, where B' and B'' are two concentric disks in B of radii $1 - \eta$ and $1 - 2\eta$, respectively, for sufficiently small $\eta > 0$. To construct $\bar{u}$, take a function φ on $[0,1]$ such that $0 \leq \varphi \leq 1$, with $\varphi = 1$ on $[0, \eta]$ and equal to zero on $[2\eta, 1]$. Now on the line $x \times R$, where the distance from x to the center 0 is r, carry the point $x \times t$ to the point which divides, in the ratio

$$\varphi(1 - r) : (1 - \varphi(1 - r)),$$

the segment between $x \times t$ and $u(x \times t)$. These homeomorphisms define the homeomorphism $\bar{u}$, which, obviously, provided only that η was chosen sufficiently small, satisfies the requirements imposed on it.

It remains for us to reduce Theorem 2 to the case just considered. Let us note first of all that the embedding q/B is locally flat. Indeed, using the fact that the embedding q/B_+ is locally flat, we can easily construct such a homeomorphism $h : S^n \to S^n$, fixed on $q(B_-)$, that

$$qB = q\breve{B}_-,$$

and thus the embedding q/B is equivalent to the embedding $q/\breve{B}_-$, i.e. is locally flat. Moreover, it is unknotted in S^n. Construct a two-sheeted covering over S^n, branched over $q\dot{B}$. Since, as was said, the embedding $q/\dot{B}$ in S^n is trivial, this branched covering is a sphere. The preimage of each of the cells B_+, B_- in the covering is a cell, covering its image two-to-one with a fold along the branch sphere. Obviously, these cells are embedded locally flatly in the covering. Moreover, they have a common boundary and, consequently, by the case analyzed above, their union is embedded locally flatly. But $q\dot{B}$ is embedded locally at the interior points of $B_+ \cap B_-$ in S^n in the same way as the union of the covering cells is embedded in the covering sphere. Thus, the embedding q is locally flat, at least at the points

$$\mathrm{Int}(B_+ \cap B_-).$$

But then it is locally flat everywhere (see the end of the proof of the theorem on the union of cells in ([4]).)

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Note: Figure translations are in progress. See original paper for figures.