SOME PROPERTIES OF POLYNOMIAL APPROXIMATIONS BY THE RITZ METHOD

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Abstract

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SOME PROPERTIES OF POLYNOMIAL APPROXIMATIONS BY THE RITZ METHOD

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1°. As a rule, approximate solutions of positive definite problems constructed by the Ritz process converge only in the energy metric. In particular, if one is dealing with a differential equation of order $2k$, then derivatives of order k converge (in the L_2 metric). Of interest are those cases in which one can assert convergence of derivatives of higher orders, in particular when the residual in the equation tends to zero. A fairly general case of this kind is indicated in ([1])*. I. K. Daugavet ([3]) found that, for an ordinary differential equation, convergence of derivatives of arbitrarily high order takes place if polynomials are taken as the coordinate functions and the solution possesses a sufficiently large number of derivatives. In the present note an analogous result will be obtained for any number of independent variables, and the differentiability requirements on the solution will be substantially reduced.

2°. We shall use the following scheme for the study of projection methods, proposed by L. V. Kantorovich ([4]) (see also ([5])).

Consider the equation

$$Au = f, \tag{1}$$

where A is a linear operator mapping some Banach space B_1 one-to-one onto the Banach space B_2, in such a way that the operators A and A^{-1} are bounded. We denote the norm in B_k, $k = 1, 2$, by $\|\cdot\|_k$. Construct a sequence of finite-dimensional subspaces $B_1^{(n)}$ of the space B_1 possessing the property that

$$E_n(u) = \inf_{u^{(n)} \in B_1^{(n)}} \|u - u_n\|_1 \longrightarrow 0, \quad \forall u \in B_1. \tag{2}$$

Put $B_2^{(n)} = AB_1^{(n)}$, and for each n choose an operator Q_n projecting B_2 onto $B_2^{(n)}$. We shall construct an approximate solution of equation (1) as an element $u_n \in B_1^{(n)}$ satisfying the equation

$$Au_n = Q_{nf}. \tag{3}$$

If u^* is the exact solution of equation (1), then

$$\|u^* - u_n\|_1 \le (1 + \|A^{-1}Q_{nA}\|)E_n(u^*);$$

hence

$$\|u^* - u_n\|_1 = O(\|Q_n\|E_n(u^*)). \tag{4}$$

3°. Let $B_1 \subset B_2 \subset H$, where the symbol $\subset$ denotes a dense embedding; assume that the spaces B_2 and H are Hilbert spaces. We denote the scalar product and norm in H by the usual symbols (,) and $\|\cdot\|$. Choose po-

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* In the arguments of article ([1]), one important detail was omitted. This omission was corrected by G. M. Vainikko ([2]).

a sequence $\{\varphi_n\}$ of coordinate elements, complete in B_1. By $B_1^{(n)}$ we shall mean the subspace of the elements $\varphi_1, \varphi_2, \dots, \varphi_n$, and the projectors Q_n are defined by the relations

$$(f - Q_n f, \varphi_j) = 0, \qquad j = 1, 2, \dots, n. \tag{5}$$

The system (5) is the Bubnov-Galerkin system for equation (1); it coincides with the Ritz system if the operator A is positive definite in H. This is the case that we shall consider below.

The norm of the projector Q_n is estimated by the inequality

$$\|Q_n\| \le C\sqrt{\Lambda_n^{(n)}/\lambda_1^{(n)}}, \qquad C = \text{const}, \tag{6}$$

where $\lambda_1^{(n)}$ and $\Lambda_n^{(n)}$ are, respectively, the smallest eigenvalue of the matrix $\|(A\varphi_j, \varphi_k)\|_{j,k=1}^{j,k=n}$ and the largest eigenvalue of the matrix $\|(A\varphi_j, A\varphi_k)\|_{j,k=1}^{j,k=n}$. If the coordinate system $\{\varphi_n\}$ is strongly minimal in the energy space of the operator A, then $\lambda_1^{(n)} \ge$ const, and

$$\|Q_n\| \le C_1\sqrt{\Lambda_n^{(n)}}; \qquad C_1 = \text{const}. \tag{7}$$

4°. Consider the equation

$$\sum_{|\alpha|+|\beta|=0}^{2k} D^{\alpha}(A_{\alpha\beta} D^{\beta} u) = f(x) \tag{8}$$

under the boundary conditions

$$D^{\alpha} u\big|_{\Gamma} = 0, \qquad |\alpha| = 0, 1, \dots, k-1. \tag{9}$$

Here x is a point of a bounded domain Ω, belonging to m-dimensional Euclidean space and homeomorphic to a ball; $u(x)$ and $f(x)$, generally speaking, are vector functions; α and β are multiindices of dimension m. We assume that the matrices $A_{\alpha\beta}$ have in $\overline{\Omega}$ sufficiently many continuous derivatives with respect to the coordinates of the point x, that equation (8) is elliptic and nondegenerate in Ω, and that the operator of the problem (8)–(9) is positive definite in $L_2(\Omega)$.

Let $\varphi(x) = 0$ be the equation of the boundary Γ of the domain Ω. We assume that the function $\varphi(x)$ is positive in Ω, that its gradient is nonzero on Γ, and that it has in $\overline{\Omega}$ sufficiently many continuous derivatives.

Finally, assume that the solution of the problem (8)–(9) is $u^* \in C^r(\overline{\Omega})$, where $r > 2k$.

Take $B_1 = \overset{\circ}{W}{}_2^{(s)}(\Omega)$, $B_2 = W_2^{(l)}(\Omega)$, $H = L_2(\Omega)$, where $s = 2k + l \le r$. As coordinate functions we take polynomials multiplied by $\varphi^k(x)$. The coordinate functions will be regarded as orthonormalized in $W_2^{(k)}(\Omega)$; this does not affect the approximate solution, but makes the coordinate system strongly minimal in the energy space of the operator A. We shall assume that the approximate solution u_n contains polynomials of degree $\le n$ in each of the Cartesian coordinates of the point x.

Using the embedding theorems of S. L. Sobolev and the estimate for the norm of the derivative of a polynomial [6], one can prove that $\Lambda_n^{(n)} = O(n^{4k+4l+1})$, and, by formula (7), $\|Q_n\| = O(n^{2k+2l+1/2})$.

Using the main theorem of the article [7], it is not difficult to find that

$$E_n(u^*) = O(n^{-r+s})\omega_r(u^*, 1/n),$$

where $\omega_r(u, \delta)$ is the largest of the moduli of continuity of the derivatives of order r of the function $u(x)$. Now, by formula (4),

$$\|u^* - u_n\|_{\overset{\circ}{W}{}_2^{(s)}} = O(n^{-r+4k+3l+1/2})\omega_r(u^*, 1/n). \tag{10}$$

In particular, convergence of the residual in equation (8) takes place if $r \geqslant 4k + {}^1/_2$.

The extension to the Bubnov–Galerkin process and to non-self-adjoint equations presents no difficulties.

For ordinary differential equations, I. K. Daugavet [3] obtained the estimate

$$\|u^* - u_n\|_{C(s)} \leqslant Cn^{-r-3k+4s+{}^1/_2-\alpha}, \qquad C = \text{const}, \tag{11}$$

under the assumption that $u^* \in C^{(r+\alpha)}$. In the case of interest to us, $s = 2k + l$, estimate (11) gives

$$\|u^* - u_n\|_{C(s)} \leqslant Cn^{-r+5k+4l+{}^1/_2-\alpha}.$$

For partial differential equations with $k = 1, 2$, V. P. Il’ in [8] obtained an estimate somewhat better than (10): in V. P. Il’ in’ s estimate the term ${}^1/_2$ is absent from the exponent.

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Note: Figure translations are in progress. See original paper for figures.