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Abstract

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NONEXISTENCE OF CERTAIN SOLVABLE EXTENSIONS

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Let L be a differential operation with constant coefficients defined on $C^\infty(R^n)$. In every domain $V \subset R^n$ with compact closure, the minimal and maximal operators generated by L in the usual way (see (^{1})) can be defined:

$$L^0,\ \tilde{L} : H(V) \to H(V),$$

where $H(V)$ is the Hilbert space $\mathcal{L}_2(V)$ of complex functions. For $u \in C^\infty(R^n)$ the restriction $u|_S$, $S = \partial V$, is defined. Let

$$\Gamma u|_S = 0 \qquad (\Gamma)$$

be some system of boundary conditions, and let

$$L_\Gamma : H(V) \to H(V)$$

be the closure in $H(V)$ of the operator that is the restriction of L to the linear manifold of functions subject to the conditions (Γ). If the conditions (Γ) are such that

$$L^0 \subset L_\Gamma \subset \tilde{L}$$

and L_Γ^{-1} exists, defined on all of H, then L_Γ is a solvable extension of the operator L^0 determined by the conditions (Γ).

In studying ways of describing solvable extensions of the operator L^0, the following question naturally arises: suppose that on some part $\check{S}$ of the boundary S a system of conditions is given

$$\check{\Gamma} u|_{\check{S}} = 0; \qquad (\check{\Gamma})$$

can it be "extended" to conditions of the form (Γ) that determine a solvable extension L_Γ? In other words, do there exist conditions (Γ) such that their restriction to $\check{S}$ is defined and this restriction coincides with $(\check{\Gamma})$?

Restrictions to $\check{S}$ of conditions (Γ) determining a solvable extension automatically give examples of cases in which the answer is positive. The purpose of this paper is to indicate the simplest situations in which one can give a negative answer. Such a negative answer shows, in a number of cases, the inevitability of using nonclassical boundary conditions ([2],[3]) in describing solvable extensions of general differential operators. As will be shown, consideration of the problem described differs essentially from checking, say, the ill-posedness of the Cauchy problem.

Let V' be a subdomain of V, representable in the form $V' = V_1 \times V_2$. Then the boundary $\partial V'$ decomposes into the parts

$$S_1 = \partial V_1 \times V_2, \qquad S_2 = V_1 \times \partial V_2.$$

We shall assume that S_1 coincides with the part $\check{S}$ of the boundary $S = \partial V$ that interests us. Let $\check{L}$ be the closure in $H(V')$ of the operator defined by restricting L to the manifold of functions in C^∞ subject to the conditions $(\check{\Gamma})$ on $\check{S} = S_1$ and to the conditions of vanishing together with all derivatives on S_2. Then the restriction to V' of any extension of the operator L^0 (on V) generated by conditions (Γ) that coincide on $\check{S}$ with $(\check{\Gamma})$ will obviously be an extension for $\check{L}$; and, in order to prove the nonexistence of an extension of the conditions $(\check{\Gamma})$ determining a solvable extension L_Γ, it is enough to establish the unboundedness of $\check{L}^{-1}$ (if $\check{L}^{-1}$ does not exist, then the conditions $(\check{\Gamma})$ are, obviously, “insufficient”).

The simplest case is when $\check{L}$ “splits,” i.e., is representable in the form $\check{L} = T - A^0$ and

$$\check{L} : H_1 \otimes H_2 \to TH_1 \otimes H_2 - H_1 \otimes A^0 H_2, \tag{1}$$

where $H_\sigma = H(V_\sigma)$, $\sigma = 1, 2$ (separation of variables, cf. (4)).

Under the assumptions made, the operator A^0 in (1) is a minimal operator in $H(V_2)$ (defined by a certain differential operation in the corresponding group of variables). From the point of view of operator theory this means that for any $\lambda \in C$ (the complex plane) the operator $(A^0 - \lambda)^{-1}$ exists, is bounded, and is defined on a non-dense subset of $H(V_2)$ (i.e., the resolvent set ρA^0 is empty). Moreover the estimate

$$\left\|(A^0 - \lambda)^{-1}\right\| \leq N < \infty \tag{2}$$

is uniform in λ. Hence it follows immediately that

Proposition 1. *If, under the assumptions made above, there exists a system of eigenfunctions of the operator T forming a Riesz basis in H_1 ([5]), then the operator $\check{L}^{-1}$ is bounded.*

Consequently, of interest to us will be operators T satisfying “irregular” ([6]) conditions.

Remark 1. Classical examples of the unboundedness of the operator L^{-1} ($\check{L} = T - A$) for an operator A such that, for every $\lambda \in C$, the operator $(A-\lambda)^{-1}$ exists and is bounded (for example, Hadamard' s proof of the ill-posedness of the Cauchy problem for the Laplace operator) are connected with the use of the dependence of N on λ in an inequality of the form (2): $N = N(\lambda) \to \infty$ under the corresponding behavior of λ.

Remark 2. The family of operators $(A^0-\lambda)^{-1}$ does not define an operator function of A^0 in the usual sense: the domain of definition of each of the operators depends essentially on λ.

To clarify the question posed, it is convenient to consider, together with T, a family of operators T_η admitting a simple spectral description: for every $\eta \neq 0, \infty$ there exists a system of eigenfunctions of the operator T_η, forming a Riesz basis $\{\varphi_\chi, \eta\}$ in H_1. Moreover, as $\eta \to 0$ ($\eta \to \infty$), $T_\eta \to T$ (in the corresponding sense), while the constants characterizing the Riesz basis tend to zero or to infinity.

Let η be fixed and $T_\eta \varphi_\chi = \chi \varphi_\chi$ (we shall not explicitly indicate the dependence of χ, φ_χ on η). From inequality (2) it follows that, under the assumptions made, one cannot establish the convergence $\|\check{L}_\eta^{-1}\| \to \infty$ as $\eta \to 0, \infty$ ($\check{L}_\eta = T_\eta - A^0$) by considering the action of $\check{L}_\eta$ on elements of the form $u_1 \varphi_\chi$, $u_1 \in H_2$ (cf. Remark 1). Therefore consider u of the form $u = u_1\varphi_\chi - u_2\varphi_\mu$, $H_2 \ni u_1, u_2$; $\chi \neq \mu$. Then

$$-\check{L}_\eta u = (A^0 - \chi)u_1\varphi_\chi - (A^0 - \mu)u_2\varphi_\mu.$$

Denoting $A^0 - \lambda = A_\lambda$ and setting

$$A_\chi u_1 = f; \qquad A_\mu u_2 = f; \qquad f \in \mathfrak{R}_{A_\chi} \cap \mathfrak{R}_{A_\mu} \tag{3}$$

($\mathfrak{R}_Q$ is the range of the operator Q); $\|\varphi_\chi\|^2 = \|\varphi_\mu\|^2 = \omega$ (it is not always convenient to take $\omega = 1$), we shall have

$$\left\|\check{L}_\eta^{-1}\right\|^2 = \sup_g \frac{\left\|\check{L}_\eta^{-1} g\right\|^2}{\|g\|^2} \geq \frac{\|u_\chi - u_\mu\|^2 + 2\operatorname{Re}\{(u_\chi, u_\mu)[(\varphi_\chi, \varphi_\mu)/\omega - 1]\}}{2\|f\|^2\left(1 - \operatorname{Re}(\varphi_\chi, \varphi_\mu)/\omega\right)}, \tag{4}$$

where $u_\chi = u_1$, $u_\mu = u_2$ are determined from (3). In view of the presence of the estimate, uniform in μ, χ,

$$\left|(u_\chi, u_\mu)\right| \leq \left\|A_\mu^{-1} f\right\| \left\|A_\chi^{-1} f\right\| \leq c\|f\|^2$$

to prove the unboundedness of relation (4) as $\eta \to 0$ ($\eta \to \infty$) it is necessary and sufficient to establish the unboundedness of

$$l(\eta|\mu,\varkappa) \equiv (\|u_\varkappa - u_\mu\| : \|f\|)^2 \left(1 - \operatorname{Re}(\varphi_\varkappa, \varphi_\mu)/\omega\right)^{-1}. \tag{5}$$

In turn, a necessary condition for the growth of (5) is, obviously, the convergence

$$\varepsilon(\eta|\mu,\varkappa) \equiv 1 - \operatorname{Re}(\varphi_\varkappa, \varphi_\mu)/\omega \to 0 \tag{6}$$

under a suitable choice of the sequences $\eta,\ \mu(\eta),\ \varkappa(\eta)$.

If V_2 is a parallelepiped, we may take $u_\mu, u_\varkappa$ of the form

$$u_\mu = P(x)e^{i\alpha x}, \qquad u_\varkappa = Q(x)e^{i\alpha x}, \qquad x \in V_2,$$

where P, Q are polynomials ensuring that $u_\mu, u_\varkappa$ belong to $\mathfrak{D}_{A^0}$ and that the equalities $A_\mu u_\mu = A_\varkappa u_\varkappa = f$ hold. We may represent $A_\mu u_\mu$ in the form

$$A_\mu u_\mu = [P_A(x,\alpha) - (A(\alpha) - \mu)P(x)]e^{i\alpha x},$$

where $P_A(x,\alpha)$ is a certain polynomial with coefficients depending on α. The corresponding representation can also be written for $A_\varkappa u_\varkappa$.

The connection between the spectrum of the operators T_η and the structure of the operator A, on which the subsequent constructions are based, is expressed by the assertion: the sequence of norms $\|\tilde{L}_\eta^{-1}\|$ for values of η belonging to the critical sequence determined by condition (6) proves to be unbounded if the sequence of roots α of the equations

$$A(\alpha) - \mu = 0 \tag{7}$$

can be chosen so that the values $|\operatorname{Re} i\alpha|$ remain bounded.

The simplest example of a family of operators T_η possessing the described properties is given by the operators $T_\eta(-iD_t)$, where $T(\nu)$ is a polynomial with constant complex coefficients $(T(-iD)e^{i\nu t} = T(\nu)e^{i\nu t})$, V_1 is the interval $0 \le t \le 2\pi$, and the boundary conditions have the form

$$D^l v\big|_{t=0} - \vartheta D^l v\big|_{t=2\pi} = 0, \qquad l = 0, 1, \dots, m-1, \tag{Γ_t}$$

where m is the order of T; $\vartheta \ge 0$ is a real parameter related to η by the relation $\eta = \ln\vartheta/2\pi$. Then, if $\nu_k = k + i\eta,\ k = 0,\ \pm1,\ \pm2, \dots$, the equalities

$$\varkappa = T(\nu_k); \qquad \varphi_\varkappa(t) = e^{i\nu_k t}; \qquad T\varphi_\varkappa = \varkappa\varphi_\varkappa$$

give an exhaustive description of the eigenfunctions and eigenvalues of the operators T_η. Moreover,

$$\omega = \|\varphi\|^2 = (1 - e^{-4\pi\eta})/2\eta; \qquad \frac{(\varphi_\mu, \varphi_\varkappa)}{\omega} = \frac{1 + i[(m-k)/2\eta]}{1 + (m-k)^2/4\eta^2}, \tag{8}$$

where $\nu_m = m + i\eta$; $\mu = T(\nu_m)$. Thus, for fixed m, k and $\eta \to \pm\infty$, condition (6) will be satisfied.

Let us consider the behavior of the first factor in (5): the ratio $\|u_\mu - u_\varkappa\|^2/\|f\|^2$. First of all, on the simplest example $A \equiv D_x$, $A(\alpha) = i\alpha$, we shall demonstrate the essential nature of the choice of α in accordance with condition (7). In this simplest case the numerator and denominator of the indicated ratio can be written in the form

$$\|(P-Q)e^{i\alpha x}\|, \qquad \|(D_x P - MP)e^{i\alpha x}\| = \|(D_x Q - KQ)e^{i\alpha x}\|,$$

respectively, where $M = \mu - i\alpha$, $K = \varkappa - i\alpha$. In this case, as is not difficult to verify, when P and Q are determined from condition (3) (ensuring fulfillment of the last identity), we obtain the dependence of $P - Q$ only on $M - K = m - k$, whereas the coefficients of the denominator depend essentially on η, and, for α independent of $\mu, \varkappa$, as $\eta \to \pm\infty$ the decrease of the first factor in (5) makes the product bounded, despite the growth of the second factor. At the same time, when choosing α satisfying the equality $\mu - i\alpha = 0$, while preserving

boundedness of $k - m$, taking into account that

$$\|x^k e^{i\alpha x}\| \sim C|\operatorname{Re} i\alpha|^{-1} e^{\operatorname{Re} i\alpha}, \tag{9}$$

we obtain, if $\operatorname{Re} i\alpha$ does not depend on η (for example, $T \equiv iD_t$), growth of expression (5) as a function of η as $\eta \to \pm\infty$. If, however, for the indicated choice of α, $\operatorname{Re} i\alpha$ turns out to depend on η (for example, $T \equiv D_t$), then again, taking into account (9) and the character of the dependence on α of the coefficients of the polynomial in the expression for f, we see that the rapid decrease of the first factor in (5) compensates the growth of the second factor, and the product remains bounded. Analogous phenomena determine the choice (7) also for more general operators A.

When the norms $\|\check{L}_\eta^{-1}\|$ grow without bound, verification of the nonexistence of a solvable extension corresponding, for example, for the family under consideration to the case $\vartheta = 0$ ($\eta = -\infty$), no longer presents difficulty: the function $\hat{u} = u_\mu(\varphi_\mu - \varphi_\mu^0) - u_\varkappa(\varphi_\varkappa - \varphi_\varkappa^0) = u - \tilde{u}$, with a corresponding choice of $\varphi_\mu^0, \varphi_\varkappa^0$, belongs, for any $\eta, \varkappa, \mu$, to $\mathfrak{D}_{\check{L}}$ (the domain of definition of the operator $\check{L}$) and satisfies homogeneous conditions at $t = 0$. Here

$$\|\check{L}^{-1}\|^2 \geq \frac{\|\hat{u}\|^2}{\|\check{L}\hat{u}\|^2} = \frac{\|u\|^2 - 2\operatorname{Re}(u, \tilde{u}) + \|\tilde{u}\|^2}{\|Lu\|^2 - 2\operatorname{Re}(Lu, \check{L}\tilde{u}) + \|\check{L}\tilde{u}\|^2} \approx \frac{\|u\|^2}{\|Lu\|^2} \to \infty$$

as $\eta \to -\infty$, since the presence of the factor $e^{-4\pi\eta}$ in $\|\varphi_{\varkappa}\|^2 = \|\varphi_{\mu}\|^2 = \omega$ makes the additional terms that arise negligible.

The family of operators T_η considered is very convenient for proving the nonexistence of solvable extensions of the operator L that leave free of conditions certain (in particular, nonempty) portions of the boundary. At the same time, already for the operator $T \equiv D_t^3$, the passage from Cauchy conditions to irregular conditions of the form $v|_{t=0} = Dv|_{t=0} = 0;\ v|_{t=2\pi} = 0$ leads, unfortunately, to very cumbersome calculations because of the complexity of the asymptotics of the eigenvalues corresponding to the regular conditions replacing the conditions (Γ_t) in this case.

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Note: Figure translations are in progress. See original paper for figures.