ON THE "HIDDEN" SYMMETRY

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ON THE "HIDDEN" SYMMETRY

OF THE RELATIVISTIC KEPLER PROBLEM

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As is well known, the spectrum of the levels of the hydrogen atom possesses an "accidental" degeneracy (the independence of the energy $\varepsilon_{nl} = -1/2n^2$ from the orbital angular momentum l). V. A. Fock showed ([1]) that the cause of this degeneracy is the symmetry of the Hamiltonian with respect to the group $O(4)$, broader than the group of geometrical symmetry of the potential $V(r) = -\alpha/r$. The generators of the symmetry group are the integrals of motion known from classical mechanics ([2]): the angular momentum $\mathbf{L}$ and the Runge–Lenz vector $\mathbf{A}$:

$$\mathbf{A} = \mathbf{n} + \frac{1}{2m\alpha}\{(\mathbf{L}\times\mathbf{p}) - (\mathbf{p}\times\mathbf{L})\}, \qquad \mathbf{n} = \mathbf{r}/r. \tag{1}$$

The Kepler problem in a space of any number of dimensions was solved by S. P. Alliluev ([3]). The symmetry for states of the continuous spectrum of the hydrogen atom has been considered in recent works ([4,5]).

Taking account of the spin–orbit interaction and other relativistic effects removes the accidental degeneracy in l. The Dirac equation in a Coulomb field admits an exact solution in hypergeometric functions, and the energy of a bound state is equal to ([6])

$$E_{nj} = m\left[1 + \frac{\alpha^2}{\left(n - |\varkappa| + \sqrt{\varkappa^2 - \alpha^2}\right)}\right]^{-1/2}; \tag{2}$$

where $\alpha = 1/137$; n is the principal quantum number;

$$\varkappa = \begin{cases} j + 1/2 = l + 1, & \text{if } j = l + 1/2, \\ -(j + 1/2) = -l, & \text{if } j = l - 1/2. \end{cases} \tag{3}$$

For states which in the nonrelativistic approximation pass into a degenerate level with principal quantum number n, $\varkappa$ takes the values $\varkappa = \pm 1, \pm 2, \dots, \pm(n-1), n$. Since E_{nj} depends only on $|\varkappa|$, all levels except the highest one ($\varkappa = n$) remain doubly degenerate.

Consequently, the Dirac equation in a Coulomb field has a symmetry somewhat greater than the geometrical symmetry of the group $O(3)$. Therefore, in addition to the usual integrals of motion (energy, total angular momentum $\mathbf{J}$, parity $P = (-1)^l$, and the Dirac operator K), there must exist an additional integral, specific to the Coulomb field. It was found explicitly by Johnson and Lippmann ([7])

$$I = \vec{\sigma}\mathbf{n} - \frac{i}{m\alpha} K \rho_1 (H - m\rho_3), \tag{4}$$

where $K = \beta(\vec{\sigma}\mathbf{L} + 1)$, $H = \vec{\alpha}\mathbf{p} + \beta m - \alpha/r$, $\vec{\alpha} = \rho_1 \vec{\sigma}$, $\beta = \rho_3$.

We shall consider, with the aid of the operator I, the dynamical symmetry group of the relativistic Kepler problem.*

* Various aspects of this problem are also the subject of works ([8–10]), in which, in particular, the properties of the operator I are studied in detail. However, the question of the hidden-symmetry group of the Dirac equation in a Coulomb field was not considered in those works.

For greater clarity it is convenient to pass to the nonrelativistic approximation. The operator I, in the usual representation of the matrices γ_μ, has the form

$$I = \begin{pmatrix} \sigma\mathbf{A} & \frac{i}{mr}(\sigma\mathbf{L} + 1) \\ -\frac{i}{mr}(\sigma\mathbf{L} + 1) & 2\sigma\mathbf{n} - \sigma\mathbf{A} \end{pmatrix}, \tag{5}$$

whence it is clear that in the nonrelativistic limit I goes over into $\sigma\mathbf{A}$ (see also (8)). In the nonrelativistic Kepler problem there are 3 vector integrals of motion: $\mathbf{L}$, $\mathbf{A}$, and $\mathbf{B} = \frac{1}{2}\{(\mathbf{L} \times \mathbf{A}) - (\mathbf{A} \times \mathbf{L})\}$, with

$$\mathbf{LA} = \mathbf{LB} = \frac{1}{2}(\mathbf{AB} + \mathbf{BA}) = 0 \tag{6}$$

(these relations give the quantum-mechanical analogue of the orthogonality of the triad of vectors $\mathbf{L}, \mathbf{A}, \mathbf{B}$). Forming 3 scalars

$$\hat{K} = \sigma\mathbf{L} + 1, \qquad \hat{I} = \sigma\mathbf{A}, \qquad \hat{I}' = \sigma\mathbf{B} \tag{7}$$

and using the commutation relations between L_i and A_j (see (4)), we find

$$[\hat{K},\hat{I}]=2i\hat{I}';\qquad [\hat{I},\hat{I}']=2i\hat{I}^2\hat{K};\qquad [\hat{I}',\hat{K}]=2i\hat{K}^2\hat{I};\tag{8}$$

$$[\hat{K},\hat{I}]_+=[\hat{I},\hat{I}']_+=[\hat{I}',\hat{K}]_+=0\tag{9}$$

(from (9) it follows that the squares $\hat{K}^2$, $\hat{I}^2$, and $\hat{I}'^2$ commute with all the operators $\hat{K},\hat{I}$, and $\hat{I}'$). The eigenvalues of K^2 are $\varkappa^2\geq 1$, i.e. K^2 is a positive operator. If I^2 does not vanish, then one can pass to the normalized operators:

$$I_1=\frac{1}{2}(\hat{I}^2)^{-1/2}\hat{I},\qquad I_2=\frac{1}{2}(\hat{K}^2\hat{I}^2)^{-1/2}\hat{I}',\qquad I_3=\frac{1}{2}(\hat{K}^2)^{-1/2}\hat{K},\tag{10}$$

for which (8) becomes the standard commutation relations for the generators of the group $SU(2)$

$$[I_i,I_j]=i\varepsilon_{ijk}I_k.\tag{11}$$

Condition (9) selects from all representations of the group $SU(2)$ only the spinor and the identity representations.* Let us now find the eigenvalues of $\hat{I}^2$. From (1) we have**

$$(\sigma\mathbf{A})\,\frac{\chi(r)}{r}\Omega_{jlM}=\frac{\varkappa}{r}\left(\frac{d}{dr}-\frac{\varkappa}{r}+\frac{1}{\varkappa}\right)\chi(r)(\sigma\mathbf{n})\Omega_{jlM},\tag{12}$$

where Ω_{jlM} is a spherical spinor ([6]). The operators that transform into one another Coulomb wave functions with the same value of n were considered by Infeld and Hull ([11]). Introducing the variable $\varkappa$ according to (3), it is not difficult to present their results in the form $(l+l'=2j,\ l-l'=\pm1)$:

$$\chi_{nl'}(r)=H_n^{(l\to l')}\chi_{nl}(r),\qquad H_n^{(l\to l')}=-\frac{n\varkappa}{\sqrt{n^2-\varkappa^2}}\left(\frac{d}{dr}-\frac{\varkappa}{r}+\frac{1}{\varkappa}\right).\tag{13}$$

Hence we find

$$\hat{K}\psi_{n\varkappa M}=\varkappa\psi_{n-\varkappa,M},$$

$$\hat{I}\psi_{n\varkappa M}=\left(1-\frac{\varkappa^2}{n^2}\right)^{1/2}\psi_{n,-\varkappa,M},\qquad \hat{I}'\psi_{n\varkappa M}=i\varkappa\left(1-\frac{\varkappa^2}{n^2}\right)^{1/2}\psi_{n,-\varkappa,M}\tag{14}$$

(we emphasize that the last two formulas have such a simple form only in the nonrelativistic case, when the ψ-function becomes two-component). For $n \neq \varkappa$ the operators $\hat{I}$ and $\hat{I}'$ permute the states $(n, \varkappa, M)$ and $(n, -\varkappa, M)$, "accidentally" degenerate in the Coulomb field. Since $\hat{K}, \hat{I}$, and $\hat{I}'$ commute with J, the group of dynamical symmetry of these levels is

* Analogous restrictions also appear in other problems with hidden symmetry. Thus, in the case of the nonrelativistic hydrogen atom the condition $\mathbf{LA} = 0$ leads to the fact ([4,5]) that, of all irreducible representations $D(j_1, j_2)$ of the group $O(4)$, on the wave functions of the discrete spectrum only representations with $j_1 = j_2 = (n-1)/2$ are realized. In the present case the additional conditions (9) turn out to be so strong that they restrict the choice of possible representations to two.

** Formulas (12) and (13) are written in the atomic system of units, i.e. $\alpha = e^2 = m = \hbar = 1$.

is $SU(2) \otimes SU(2)$ (the first of the factors describes the degeneracy with respect to the projection of the total angular momentum J, the second the twofold "accidental" degeneracy). A special situation arises for the upper level ($\chi = n$), when $K = n$, and $I = I' = 0$, and the transition to the normalized operators (10) becomes impossible. Since for this state not all generators vanish, it cannot be regarded as the unit representation of the group $SU(2)$. Therefore, strictly speaking, the group of dynamical symmetry of the relativistic Kepler problem is $SU(2) \otimes G$, where G is defined abstractly, by means of the relations (8). In this case, for all representations in which I and I' do not vanish, G is isomorphic to the group $SU(2)$.

To clarify the structure of the group G, it was sufficient to consider the operator I in the nonrelativistic limit. To obtain the splitting of the levels, it is necessary to take into account the next approximation. Retaining in (5) terms $\sim \alpha^2$, we have

$$I = \vec{\sigma}\mathbf{A} + \frac{1}{4m}\{V(\vec{\sigma}\mathbf{A}) + (\vec{\sigma}\mathbf{A})V - 2\alpha V(\vec{\sigma}\mathbf{n})\} + \dots, \qquad V = -\frac{\alpha}{r}, \tag{15}$$

whence we find the relativistic correction to (14):

$$I\psi_{n\chi M} = \left(1 - \alpha^2 \frac{|\chi|}{2n^2(n+|\chi|)}\right)\sqrt{1 - \frac{\chi^2}{n^2}}\,\psi_{n,-\chi,M}. \tag{16}$$

On the other hand, from (4) there follows the operator relation

$$I^2 = 1 + \frac{K^2}{\alpha^2 m^2}(H^2 - m^2) \quad \text{or} \quad H = m\sqrt{1 - \alpha^2\frac{1-I^2}{K^2}}, \tag{17}$$

which is exact. Substituting into (17) the expansion (16) for I and retaining terms of order α^4, we obtain from this the well-known formula for the energy of the level

$$\varepsilon_{nj} = \frac{E_{nj}}{m} - 1 = -\frac{\alpha^2}{2n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{|\chi|} - \frac{3}{4}\right) + ...\right]. \tag{18}$$

Thus, the splitting of the level when relativistic interactions are included can be found from group-theoretical considerations.

We now show that the conclusions drawn above about the structure of the group G are not connected with the nonrelativistic approximation and are exact. It is known ([7]) that also in the relativistic case the operators K and I anticommute: $[K, I]_+ = 0$.

Introducing the operator I' in a formal way,

$$I' = \frac{1}{2i}[K, I] = -iKI = iIK, \tag{19}$$

one can verify purely algebraically that the relations (8), (9) still hold. Indeed,

$$[I, I']_+ = i(I^2K + IKI) = 0, \qquad [I', K]_+ = i(IK^2 + KIK) = 0, \tag{20}$$

i.e. the relations (9) are satisfied. Hence we have $K^2 I = -KIK = IK^2$, i.e. $[K^2, I] = 0$. In an analogous way one verifies that the squares K^2, I^2, and ${I'}^2$ commute with each of the three operators K, I, and I'. After this, checking the commutation relations (8) presents no difficulty.

Although such an approach is quite rigorous, the introduction of the new operator I' here looks somewhat debatable. At the same time, if one starts from the nonrelativistic case, then the introduction of the three operators (7) is quite natural and is based on the existence of three vector integrals of the motion **L**, **A**, and **B** in the nonrelativistic Kepler problem (when spin is added, the operator $\vec{\sigma}$ is also conserved in the nonrelativistic case).

In conclusion, we note that the condition $I\psi = 0$, valid for the nondegenerate levels with $j = n - \frac{1}{2}$, $\chi = n$, uniquely determines the wave

function of this state. Writing the ψ-function in the form (6)

$$\psi_{njM} = \begin{pmatrix} ig(r)\Omega_{jlM}(n) \\ -f(r)\Omega_{jl'M}(n) \end{pmatrix}, \qquad l + l' = 2j \tag{21}$$

and using (5), (12), we find from the condition $I\psi = 0$

$$\left(\frac{\partial}{\partial r} - \frac{n}{r} + \frac{1}{n}\right)\varphi = \frac{\alpha}{r}\chi, \qquad \text{where } g = \frac{\varphi(r)}{r}, \quad f = \frac{\chi(r)}{r} \tag{22}$$

(see also (12)).

Combining (22) with the radial Dirac equation* and taking into account that

$$E_{n,n-1/2} = m\sqrt{1-\alpha^2/n^2}, \tag{23}$$

we obtain that the functions f and g for the states under consideration are proportional to one another:

$$f_{n,n-1/2}(r) = C_n g_{n,n-1/2}(r), \qquad C_n = -\sqrt{(m-E_{n,n-1/2})/(m+E_{n,n-1/2})}. \tag{24}$$

Substituting this into (22), we find:

$$\left(\frac{\partial}{\partial r} - \frac{\nu}{r} + \frac{1}{n}\right)\varphi_{n,n-1/2}(r) = 0, \tag{25}$$

where

$$\nu = n + \alpha C_n = \sqrt{n^2-\alpha^2}. \tag{26}$$

The solution of equation (25) has the form $\varphi_{n,n-1/2}(r) = \mathrm{const}\cdot e^{-r/n} r^{\nu}$. It is interesting that the exponent of the exponential in $\varphi_{n,n-1/2}(r)$ remains the same as in the nonrelativistic case.

Proof-correction note. As has become known to us, the question of the group of "hidden" symmetries of the Dirac equation in a Coulomb field is also discussed in note ([13]).

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CITED LITERATURE

[1] V. A. Fock, *Zs. f. Phys.*, **98**, 145 (1935).
[2] V. Bargmann, *Zs. f. Phys.*, **99**, 576 (1935).
[3] S. P. Alliluev, *ZhETF*, **33**, 200 (1957).
[4] A. M. Perelomov, V. S. Popov, *ZhETF*, **50**, 179 (1966).
[5] M. Bander, C. Itzykson, *Rev. Mod. Phys.*, **38**, 330, 346 (1966).
[6] A. I. Akhiezer, V. B. Berestetskii, *Quantum Electrodynamics*, M., 1959.
[7] M. H. Johnson, B. A. Lippmann, *Phys. Rev.*, **78**, 329 (1950).
[8] L. C. Biedenharn, *J. Math. Phys.*, **2**, 433 (1961); *Phys. Rev.*, **126**, 845 (1962).

[9] L. C. Biedenharn, N. V. V. Swamy, *Phys. Rev.*, **B133**, 1353 (1964).
[10] D. M. Fradkin, *Phys. Rev.*, **B135**, 1085 (1964).
[11] L. Infeld, T. E. Hull, *Rev. Mod. Phys.*, **23**, 21 (1951).
[12] P. C. Martin, R. J. Glauber, *Phys. Rev.*, **109**, 1307 (1958).
[13] I. A. Malkin, V. I. Man' ko, *JETP Letters*, **7**, 105 (1968).

* See formula (12.7) in ([6]). Note that the definition of χ in ([6]) differs from ours by a sign.

Note: Figure translations are in progress. See original paper for figures.