On Pulse Control Systems with Width Modulation

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Abstract

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On Pulse Control Systems with Width Modulation

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Control systems with pulse-width modulation (p.-w.m.), despite their great applied importance, have been little studied theoretically. Stability conditions for systems with p.-w.m. were obtained only quite recently in the works of A. Kh. Gelig ([1]), V. M. Kuntsevich and Yu. M. Chekhovoi ([2]). One special case was considered by E. Jury and B. Lee in ([3]). Below it will be shown that the general quadratic stability criterion established in ([4,5]) can, after small refinements (see Theorems 1 and 2 below), be applied to systems with p.-w.m. This makes it possible, in contrast to ([1,2]), to obtain frequency-domain stability conditions at once for systems containing, in addition to pulse modulators, also nonlinear or linear nonstationary blocks of the usual types (see Section 4°). A stability criterion for a system with one pulse modulator is established by Theorem 3.

1°. A large class of control systems (in particular, systems with p.-w.m.) is described by the equations

$$\sigma_t = \alpha_t + \int_0^t \Omega(t-\tau)\varphi_\tau d\tau - R\varphi_t, \qquad \varphi_t = \varphi[t, \sigma_\tau|_{\tau=0}^t, \psi_0]. \tag{1}$$

The first equation (1) describes the linear part of the system, and the second the nonlinear part. In (1) $\alpha_t, \sigma_t, \varphi_t$ are vector functions of orders m, m, and n, respectively; $\Omega(t)$ is a rectangular matrix function of order $n \times m$ of the impulse characteristics of the linear part of the system; R is a constant $n \times m$ matrix of the so-called tachometric feedback coefficients. In many cases $R = 0$. All quantities in (1) are real. The second equation (1) means that the outputs of the nonlinear blocks (the components of the vector φ_t) may depend on the values of the inputs (the components of the vector σ_τ) at all preceding instants of time $0 \le \tau \le t$, and also, possibly, on some vector parameter ψ_0.

The dependence of φ_t on $\sigma_\tau|_{\tau=0}^t$ and ψ_0 occurs for nonlinearities of hysteresis type. For systems with p.-w.m. some of the components of the vector function $\varphi[t, \sigma_\tau|_{\tau=0}^t, \psi_0]$ depend only on $\sigma_\tau|_{\tau=0}^t$, and this dependence has the special form indicated below (see 3°).

Below it is assumed that a solution of the system (1) exists on $(0, \infty)$.* We shall consider the condition

$$\int_0^{t_k} F(\varphi_t, \sigma_t)\, dt \geq -\gamma, \qquad t_k \to \infty. \tag{2}$$

Here φ_t, σ_t is a solution of the system (1); $F(\varphi_t, \sigma_t)$ is some real quadratic form of its arguments, and t_k is some unboundedly increasing sequence of times, $t_0 = 0$. We extend $F(\varphi_t, \sigma_t)$, preserving Hermiticity, to complex values of φ_t, σ_t (this extension is unique).

Theorem 1. *Suppose that:* (a_1) *the linear part of the system (1) is stable in the following sense:* $|\alpha_t| \in L_2(0, \infty)$, $|\Omega(t)| \leq Ce^{-\varepsilon t}$, $\varepsilon > 0$;

* If the second equation (1) describes only pulse-width modulators (their equations have the form (6)—see below), then the solution obviously exists on $(0, \infty)$. We note that in ([4]) only local existence of a solution is assumed.

(b_1) either $|\varphi_t| \leqslant \text{const}$, or $F(0, \sigma_t) \geqslant 0$ for all σ_t. Define the matrix of frequency characteristics of the linear part of the system by the formula

$$\chi(i\omega) = R - \int_0^{\infty} \Omega(t) e^{-i\omega t}\, dt \tag{3}$$

and put $\widetilde{F}(i\omega, \widetilde{\varphi}) = F(\widetilde{\varphi}, \tilde{\sigma})$, where $\tilde{\sigma} = -\chi(i\omega)\widetilde{\varphi}$, and $\widetilde{\varphi}$ is a complex vector of order n. Suppose that: (c_1) the form $\widetilde{F}(i\omega, \widetilde{\varphi})$ is a negative definite form in $\widetilde{\varphi}$ for all $-\infty \leqslant \omega \leqslant +\infty$. Then: (A_1) $|\varphi_t| \in L_2(0, \infty)$, $|\sigma_t| \in L_2(0, \infty)$; (B_1) for some constants $\delta > 0$, $\varkappa > 0$, independent of a_t, the estimate* holds

$$\delta \{ \|\varphi_t\|^2 + \|\sigma_t\|^2 \} \leqslant \gamma + \varkappa \|a_t\|^2.$$

Remark. The condition $|\Omega(t)| \leqslant Ce^{-\varepsilon t}$, $\varepsilon > 0$, can be weakened. In the case when $F(0, \sigma) \geqslant 0$, it is sufficient that $|\Omega(t)| \in L(0, \infty)$; in the case when $|\varphi_t| \leqslant \text{const}$, it is sufficient that $|\Omega(t)| \in L(0, \infty)$,

$$\int_t^{\infty} |\Omega(\tau)|\, d\tau \in L_2(0, \infty).$$

The proof of Theorem 1 for the case when $F(0, \sigma) \geqslant 0$ repeats word for word the proof of Theorem 1 in ([4]). In doing so one should take $T = t_k$. Let $|\varphi_t| \leqslant \text{const}$. Assertion (A_1) is proved word for word in the same way as Theorem 2 in ([4])**. To prove assertion (B_1), one should in formula (4.8) of ([4]) pass to the limit $T = t_k \to \infty$ and use the fact that $\gamma_0 = \gamma$ for $t_0 = 0$ and $J \leqslant 0$.

2°. Let $R = 0$ in equations (1). Suppose that condition (2) is replaced by the more general condition

$$\int_0^{t_k} F(\varphi_t, \sigma_t, \dot{\sigma}_t)\, dt \geqslant -\gamma, \qquad t_k \to \infty, \tag{4}$$

where t_k is, as before, an unbounded increasing sequence, $t_0 = 0$, and $F(\varphi_t, \sigma_t, \dot{\sigma}_t)$ is a Hermitian form of its arguments. Put

$$\widetilde{F}(i\omega, \widetilde{\varphi}) = F(\widetilde{\varphi}, \tilde{\sigma}, i\omega\tilde{\sigma}), \quad \text{where } \tilde{\sigma} = -\chi(i\omega)\widetilde{\varphi} \tag{5}$$

and $\chi(i\omega)$ is determined from (3) for $R = 0$.

Theorem 2. Let the conditions (a_1), (c_1) of Theorem 1 be satisfied (where $\widetilde{F}(i\omega, \widetilde{\varphi})$ is determined from (5)), and also: (a_2) $|\dot{a}_t| \in L(0, \infty)$, $|\dot{\Omega}(t)| \leqslant Ce^{-\varepsilon t}$, $\varepsilon > 0$; (b_2) either $F(0, \sigma_t, \dot{\sigma}_t) \geqslant 0$ for all possible $\sigma_t, \dot{\sigma}_t$, or $|\varphi_t| \leqslant$ const. Then: (A_2) $|\varphi_t| \in L_2(0, \infty)$, $|\sigma_t| \in L_2(0, \infty)$, $|\dot{\sigma}_t| \in L_2(0, \infty)$, and, consequently, $|\sigma_t| \to 0$ as $t \to \infty$; (B_2) the estimate

$$\delta\{\|\varphi_t\|^2 + \|\sigma_t\|^2 + \|\dot{\sigma}_t\|^2\} \leqslant \gamma + \varkappa\{\|a_t\|^2 + \|\dot{a}_t\|^2\}$$

holds, where the constants $\varkappa > 0$, $\delta > 0$ do not depend on a_t; in particular, $\max_{t \geqslant 0} |\sigma_t| \to 0$ if $\gamma \to 0$, $\|a_t\| \to 0$, $\|\dot{a}_t\| \to 0$.

The proof of Theorem 2 is carried out by means of the device set forth in § 5 of [4]. System (1) is reduced to system (5.1), (5.2) of [4], to which Theorem 1 is applied.

3°. The use of Theorems 1, 2 for a system with P.W.M. is based on the lemma given below, which establishes, for the pulse-width modulator, a relation of the form (2). If the system, besides the pulse-width modulator, contains a number of nonlinear blocks of ordinary types, then, as shown in [4,5], quadratic relations of the form

$$F_j = (\varphi_t, \sigma_t, \dot{\sigma}_t) = 0, \quad j = 1, \dots, p,$$

$$\int_0^{t_k} F_j(\varphi_t, \sigma_t, \dot{\sigma}_t)\, dt \geqslant -\gamma_j, \quad j = p+1, \dots, p+q.$$

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* By $|\varphi_t|$ is denoted the square root of the sum of the squares of the components of the vector φ_t, and

$$\|\varphi_t\|^2 = \int_0^{\beta} |\varphi_t|^2\, dt.$$

** The assumption $|a_t| \leqslant$ const in Theorem 2 of [4] is superfluous.

Here t_k is any sequence, $t_k \to \infty$, and γ_j are certain numbers. Among the last couplings one may also include a coupling of the form (2) for a pulse-width modulator. Setting

$$F(\varphi_t, \sigma_t, \dot{\sigma}_t) = \sum_{j=1}^{p+q} \tau_j F_j,$$

where τ_j are arbitrary for $j = 1, \dots, p$ and $\tau_j \geq 0$ for $j = p+1, \dots, p+q$, we find that (4) is satisfied. Theorem 2 or 1 will give frequency-domain stability conditions. In doing so one should, if possible, find the "envelope" of these conditions over all possible τ_j of the indicated form (see the examples in (5)). One can proceed in a completely analogous way when there are several modulators in the system. Let us derive relations (2) for a pulse-width modulator. Let $\sigma_t = \sigma(t)$, and let φ_t be the scalar input and output of the modulator; Δ is the dead-zone magnitude, and t_k is the instant at which the k-th pulse is sent. The operation of the modulator is described by the equations

$$\sigma^{(k)} = \sigma(t_k), \qquad t_{k+1} = \Psi(t_k, \sigma^{(k)}),$$

$$\varphi_t = 0 \quad \text{for } t_k \leq t < t_{k+1}, \text{ if } |\sigma^{(k)}| < \Delta,$$

$$\varphi_t = s_k(t) \quad \text{for } t_k \leq t < t_{k+1}, \text{ if } |\sigma^{(k)}| \geq \Delta. \tag{6}$$

The function $s_k(t)$ determines the shape of the k-th pulse and depends in some way on $\sigma^{(k)}$. In the case of a rectangular pulse we have

$$s_k(t) = \operatorname{sign} \sigma^{(k)} \quad \text{for } t_k \leq t < t_k + T(|\sigma^{(k)}|),$$

$$s_k(t) = 0 \quad \text{for } t_k + T(|\sigma^{(k)}|) \leq t < t_{k+1}, \tag{7}$$

where $T(\sigma)$ is some monotone and bounded function.

Lemma. Let the input σ_t and the output φ_t of the modulator with equations (6) be related by the relation

$$\sigma_t = \xi(t) + \eta(t) + a\varphi_t,$$

where the functions $\xi(t)$ and $\eta(t)$ satisfy the conditions

$$|\xi(t)| \leq b, \qquad |\eta(t)| \in L(0, \infty).$$

Denote

$$S_k(t) = \int_t^{t_{k+1}} s_k(t)\,dt, \qquad S_k = S(t_k), \qquad Q_k = \int_{t_k}^{t_{k+1}} |s_k(t)|\,dt, \qquad M_k = \int_{t_k}^{t_{k+1}} [s_k(t)]^2\,dt$$

and suppose that $|S_k(t)| \le c$ for all k, $\sigma^{(k)}$, and $t_k \le t \le t_{k+1}$, and

$$M_k^{-1}[aS_{k/2}^2 + \sigma^{(k)} S_k - bQ_k] \ge \nu \quad \text{when } |\sigma^{(k)}| \ge \Delta, \tag{8}$$

where ν is a constant independent of k. Then, for any t_k,

$$\gamma = C \int_0^\infty |\eta(t)|\,dt$$

satisfies (2) with the form $F = (\sigma_t - \nu\varphi_t)\varphi_t$.

From (8) we obtain that, in the case of rectangular pulses (7), the value of ν is found from the formula

$$\nu = \inf_{\sigma \ge \Delta} \left[\sigma + \tfrac{1}{2}(a-b)T(\sigma)\right].$$

Proof. Denote

$$I_k = \int_{t_k}^{t_{k+1}} (\sigma_t - \nu\varphi_t)\varphi_t\,dt.$$

When $|\sigma^{(k)}| < \Delta$, we have $I_k = 0$. Let $|\sigma^{(k)}| \ge \Delta$. Integrating by parts, we find

$$I_k = aS_{k/2}^2 + \sigma^{(k)} S_k - \nu M_k + \int_{t_k}^{t_{k+1}} (\xi + \eta) S_k(t)\,dt.$$

If (8) is fulfilled, we have

$$I_k \ge -c \int_{t_k}^{t_{k+1}} |\eta|\,dt.$$

Summing these inequalities, we obtain that (2) is fulfilled with the indicated values of F and γ.

Theorem 3. Consider system (1), where all quantities are scalar ($m = n = 1$), $R = 0$, and the second equation (1) has the form (6). Let the pulses of the pulse-width modulator be bounded:

$$|s_k(t)| \le c_0, \qquad |S_k(t)| \le c.$$

Let the conditions (a_1), (a_2) of Theorems 1, 2 be satisfied, and also $|a_t| \in L(0,\infty)$.

Define ν from relation (8), where

$$a = \Omega(0), \qquad b = c_0 \int_0^\infty |d\Omega(t)|.$$

Suppose that $\nu + \operatorname{Re}\chi(i\omega) > 0$ for $-\infty \le \omega \le +\infty$, where $\chi(i\omega)$ is defined from (3). Then assertion (A_2) of Theorem 2 is valid, as is the estimate

$$\|\varphi_t\|^2 + \|\sigma_t\|^2 + \|\dot{\sigma}_t\|^2 \le \chi_1 \int_0^\infty |a_t|\, dt + \chi_2(\|\alpha_t\|^2 + \|\dot{\alpha}_t\|^2),$$

where χ_j do not depend on a_t.

Proof. From (1) it follows that

$$\dot{\sigma}_t = \xi(t) + \eta(t) + a\varphi_t,$$

$\eta(t) = \alpha_t,\ |\xi(t)| \le b$ for the values of a and b indicated in Theorem 3. Applying the lemma and Theorem 2, we obtain the assertion of Theorem 3.

4°. In a completely analogous way one can obtain frequency conditions for the stability of a system with several modulators and, possibly, with several nonlinear blocks of the usual type. Consider, for example, a single-loop system with one modulator and one nonlinear block, which is described by the scalar equations

$$\sigma_{1t} = a_{1t} + \int_0^t \Omega_1(t-\tau)\varphi_{2\tau}\, d\tau, \qquad \sigma_{2t} = a_{2t} + \int_0^t \Omega_2(t-\tau)\varphi_{1\tau}\, d\tau - \rho\varphi_{1t}, \quad (9)$$

where α_{jt}, Ω_{jt} satisfy condition (a_1) of Theorem 1 and $|a_{1t}| \in L(0,\infty)$. Let $\varphi_{1t} = \varphi_1(\sigma_{1\tau}\ |_{\tau=0}^t)$ be a pulse modulator with equations (6) and bounded pulses (see Theorem 3), and let $\varphi_{2t} = \Phi(\sigma_{2t}, t)$, where $\Phi(\sigma_2, t)$ is a function satisfying the conditions

$$|\Phi(\sigma_2, t)| \le \Phi_0, \qquad 0 \le \frac{\Phi(\sigma_2, t)}{\sigma_2} \le \mu_0, \quad \sigma_2 \ne 0.$$

From the first equation (9) we find that the hypothesis of the lemma is fulfilled for

$$\xi(t) = \Omega_1(0)\varphi_{2t} + \int_0^t \dot{\Omega}(t-\tau)\varphi_{2\tau}\, d\tau,$$

$$\eta(t) = \dot{a}_{1t}, \qquad a = 0, \qquad b = \Phi_0 \left\{ |\Omega_1(0)| + \int_0^\infty |d\Omega_1(t)| \right\}.$$

Using the lemma, we obtain that (2) is fulfilled, where

$$F(\varphi_t, \sigma_t) = \tau_1(\sigma_{1t} - \nu\varphi_{1t})\varphi_{1t} + \tau_2(\sigma_{2t} - \mu_0^{-1}\varphi_{2t})\varphi_{2t},$$

and the number ν is determined from (8) for the indicated values of a and b. Theorem 1 gives the following result. If for some numbers $\tau_1 > 0$, $\tau_2 > 0$ and all $-\infty \leq \omega \leq +\infty$ the condition

$$\nu\mu_0^{-1}\tau_1\tau_2 > |\tau_1\chi_1(i\omega) + \tau_2\chi_2(i\omega)|^2$$

is fulfilled, then system (9) is stable in the following sense:

$$\varphi_{jt} \in L_2(0,\infty), \qquad \sigma_{jt} \in L_2(0,\infty)$$

and

$$\|\varphi_{jt}\| \to 0, \qquad \|\sigma_{jt}\| \to 0, \quad \text{if } \|\alpha_{jt}\| \to 0, \quad j = 1,2, \qquad \int_0^\infty |a_{1t}|\, dt \to 0.$$

Here

$$\chi_1(i\omega) = -\int_0^\infty \Omega_1(t)e^{-i\omega t}\, dt, \qquad \chi_2(i\omega) = \rho - \int_0^\infty \Omega(t)e^{-i\omega t}\, dt.$$

It is easy to obtain (see (6), p. 53, § 8) the necessary and sufficient conditions for the existence of the indicated numbers τ_1, τ_2. If $\rho = 0$, the conditions (a_1), (a_2) of Theorems 1, 2 are fulfilled, as well as the frequency condition indicated above, then according to Theorem 2 system (9) is stable in a stronger sense: assertions (A_2), (B_2) of Theorem 2 are valid with

$$\gamma = c\int_0^\infty |a_{1t}|\, dt.$$

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Note: Figure translations are in progress. See original paper for figures.