# DIFFUSION OF RADIATION IN A MEDIUM OF LARGE OPTICAL THICKNESS WITH ANISOTROPIC SCATTERING

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## Abstract

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# DIFFUSION OF RADIATION IN A MEDIUM OF LARGE OPTICAL THICKNESS WITH ANISOTROPIC SCATTERING

In the study of planetary atmospheres and water basins one encounters the problem of the diffusion of radiation in a plane layer of large optical thickness illuminated by parallel rays. For the case of a spherical scattering indicatrix this problem was solved earlier ([1]). Now, by the same method, it is solved for the case of anisotropic scattering of light. In the first part of the present note the radiation field in the deep layers of a semi-infinite medium is determined; in the second, asymptotic formulas are derived for the intensity of radiation emerging from a plane layer of large optical thickness.

1. Let parallel rays, making an angle $\arccos\zeta$ with the normal, fall on a semi-infinite medium and produce an illumination of the area perpendicular to them equal to $\pi S$. Denote by $I(\tau,\eta,\zeta)$ the azimuth-averaged intensity of the diffuse radiation traveling at optical depth $\tau$ at an angle $\arccos\eta$ to the normal, and by $B(\tau,\eta,\zeta)$ the azimuth-averaged "source function." As is known, these quantities are related by the equations:

$$\eta\,dI(\tau,\eta,\zeta)/d\tau = -I(\tau,\eta,\zeta) + B(\tau,\eta,\zeta), \tag{1}$$

$$B(\tau,\eta,\zeta) = \frac{\lambda}{2}\int_{-1}^{+1} p(\eta,\eta')I(\tau,\eta',\zeta)\,d\eta' + \frac{\lambda}{4}Sp(\eta,\zeta)e^{-\tau/\zeta}; \tag{2}$$

where $\lambda$ is the photon survival probability in an elementary act of scattering,

$$p(\eta,\eta') = \frac{1}{2\pi}\int_0^{2\pi}\chi\left(\eta\eta' + \sqrt{(1-\eta^2)(1-\eta a'^2)}\cos\varphi\right)d\varphi, \tag{3}$$

$\chi(\cos\gamma)$ is the scattering indicatrix.

From (1) and (2), with the boundary condition $I(0,\eta,\zeta) = 0$ for $\eta > 0$, we obtain an integral equation for determining the source function:

$$\begin{aligned} B(\tau, \eta, \zeta) = {} & \frac{\lambda}{2} \int_0^1 p(\eta, \eta')\, d\eta' \int_0^\tau B(\tau', \eta', \zeta) e^{-(\tau - \tau')/\eta'} \frac{d\tau'}{\eta'} \\ & + \frac{\lambda}{2} \int_0^1 p(\eta, -\eta')\, d\eta' \int_\tau^\infty B(\tau', -\eta', \zeta) e^{-(\tau' - \tau)/\eta'} \frac{d\tau'}{\eta'} \\ & + \frac{\lambda S}{4} p(\eta, \zeta) e^{-\tau/\zeta}. \end{aligned} \tag{4}$$

For the deep layers of a semi-infinite medium we have

$$B(\tau, \eta, \zeta) = Sc(\zeta) b(\eta) e^{-k\tau}, \tag{5}$$

$$I(\tau, \eta, \zeta) = Sc(\zeta) i(\eta) e^{-k\tau}, \tag{6}$$

where

$$i(\eta) = b(\eta)/(1 - k\eta). \tag{7}$$

In this case equation (4) can be written in the form

$$B(\tau, \eta, \xi) = \frac{\lambda}{2} \int_0^1 p(\eta, \eta')\, d\eta' \int_{-\infty}^\tau B(\tau', \eta', \xi) e^{-(\tau - \tau')/\eta'} \frac{d\tau'}{\eta'} + \frac{\lambda}{2} \int_0^1 p(\eta, -\eta')\, d\eta' \int_\tau^\infty B(\tau', -\eta', \xi) e^{-(\tau' - \tau)/\eta'} \frac{d\tau'}{\eta'} \tag{8}$$

Substituting (5) into (8), we arrive at an equation for determining the function $b(\eta)$

$$b(\eta) = \frac{\lambda}{2} \int_{-1}^{+1} p(\eta, \eta') \frac{b(\eta')}{1 - k\eta'}\, d\eta'. \tag{9}$$

The quantity $k$ is found from the solvability condition for this equation. For normalizing $b(\eta)$ one may use the condition

$$\frac{1}{2} \int_{-1}^{+1} b(\eta)\, d\eta = 1. \tag{10}$$

Equation (9) was first obtained by V. A. Ambartsumian ($^2$). It was subsequently solved in works by the author ($^3$), Kuper ($^4$), and others. However, as far as we know, the function $c(\xi)$ has not yet been determined. To determine it, let us differentiate both sides of equation (4) with respect to $\tau$ and compare the result obtained with equation (4). Doing this, we find

$$B'(\tau,\eta,\xi) = -\frac{1}{\xi}B(\tau,\eta,\xi) + \frac{2}{S}\int_0^1 B(\tau,\eta,\eta')B(0,\eta',\xi)\frac{d\eta'}{\eta'}. \tag{11}$$

Substitution of (5) into (11) gives

$$c(\xi) = \frac{\xi}{1-k\xi}\,\frac{2}{S}\int_0^1 c(\eta)B(0,\eta,\xi)\frac{d\eta}{\eta}. \tag{12}$$

The quantity $B(0,\eta,\xi)$ entering (12) can be found from equation (4). Setting $\tau = 0$ in it, we obtain

$$B(0,\eta,\xi) = \frac{\lambda}{4}S\left[2\xi\int_0^1 p(\eta,-\eta')\rho(\eta',\xi)\,d\eta' + p(\eta,\xi)\right], \tag{13}$$

where $\rho(\eta,\xi)$ denotes the brightness coefficient of a semi-infinite medium, defined by the formula

$$\int_0^\infty B(\tau,-\eta,\xi)e^{-\tau/\eta}\frac{d\tau}{\eta} = S\rho(\eta,\xi)\xi. \tag{14}$$

Comparing equations (12) and (13) with the equations obtained in the work of V. A. Ambartsumian ($^5$), we see that the function $c(\xi)$ can be represented in the form

$$c(\xi) = u(\xi)\xi, \tag{15}$$

where $u(\eta)$ is the intensity of radiation diffusely transmitted by a semi-infinite medium at an angle $\arccos\eta$ to the normal. Previously the function $u(\eta)$ was determined only up to a constant factor. We shall now find this factor for the case of deep layers of a semi-infinite medium. To do this, we apply the following device. Substitute in (8), instead of

Replace, in $B(\tau,\eta,\xi)$, the function by $b(\eta)e^{-\kappa\tau}$ and write the identity obtained in the form

$$\begin{aligned} b(\eta)e^{-\kappa\tau} &= \frac{\lambda}{2}\int_0^1 p(\eta,\eta')\,d\eta'\int_0^\tau b(\eta')e^{-\kappa\tau'-(\tau-\tau')/\eta'}\frac{d\tau'}{\eta'} \\ &\quad + \frac{\lambda}{2}\int_0^1 p(\eta,-\eta')\,d\eta'\int_\tau^\infty b(-\eta')e^{-\kappa\tau'-(\tau'-\tau)/\eta'}\frac{d\tau'}{\eta'} \\ &\quad + \frac{\lambda}{2}\int_0^1 p(\eta,\eta')e^{-\tau/\eta'}i(\eta')\,d\eta'. \end{aligned} \tag{16}$$

Comparing (16) with (4), we have

$$b(\eta)e^{-\kappa\tau} = \frac{2}{S}\int_0^1 B(\tau,\eta,\eta')i(\eta')\,d\eta'. \tag{17}$$

Relation (17) is valid for arbitrary $\tau$. Substituting expression (5) into (17) and using formula (15), we obtain

$$2\int_0^1 u(\eta)i(\eta)\eta\,d\eta = 1. \tag{18}$$

Formula (18) also makes it possible to find the unknown constant factor in the function $u(\eta)$. Thus the problem of determining the radiation field in deep layers of a semi-infinite medium is completely solved.

2. Let us now consider the diffusion of radiation in a plane layer of large optical thickness $\tau_0$. As before, suppose that the layer is illuminated by parallel rays.

We shall be interested in the intensities of the radiation emerging from the layer, or in the corresponding brightness coefficients $\rho(\eta,\xi,\tau_0)$ and $\sigma(\eta,\xi,\tau_0)$, related to the intensities by the relations

$$I(0,-\eta,\xi,\tau_0) = S\rho(\eta,\xi,\tau_0)\xi, \qquad I(\tau_0,\eta,\xi,\tau_0) = S\sigma(\eta,\xi,\tau_0)\xi. \tag{19}$$

In the case under consideration (for $\tau_0 \gg 1$), the results obtained above for deep layers of a semi-infinite medium can be used to determine the brightness coefficients. Let us make, mentally, a cut in the semi-infinite medium at a large optical depth $\tau_0$. Then it may be assumed that a plane layer of optical thickness $\tau_0$ is illuminated from above by parallel rays, and from below by diffuse radiation, whose intensity is determined by formula (6) at $\tau = \tau_0$ and $\eta < 0$.

On the basis of what has been said, we have the following relations:

$$\rho(\eta,\xi) = \rho(\eta,\xi,\tau_0) + 2u(\xi)e^{-\kappa\tau_0}\int_0^1 \sigma(\eta,\eta',\tau_0)i(-\eta')\eta'\,d\eta', \tag{20}$$

$$u(\xi)i(\eta)e^{-\kappa\tau_0} = \sigma(\eta,\xi,\tau_0) + 2u(\xi)e^{-\kappa\tau_0}\int_0^1 \rho(\eta,\eta',\tau_0)i(-\eta')\eta'\,d\eta'. \tag{21}$$

Here, as above, $\rho(\eta,\xi)$ denotes the brightness coefficient of the semi-infinite medium, assumed known.

From equations (20) and (21) we easily obtain

$$\rho(\eta, \xi, \tau_0) = \rho(\eta, \xi) - \frac{u(\eta)u(\xi)}{1 - N^2 e^{-2\kappa\tau_0}} M N e^{-2\kappa\tau_0}, \tag{22}$$

$$\sigma(\eta, \xi, \tau_0) = \frac{u(\eta)u(\xi)}{1 - N^2 e^{-2\kappa\tau_0}} M e^{-\kappa\tau_0}, \tag{23}$$

where

$$M u(\eta) = i(\eta) - 2\int_0^1 \rho(\eta, \eta') i(-\eta') \eta' \, d\eta', \tag{24}$$

$$N = 2\int_0^1 u(\eta) i(-\eta) \eta \, d\eta. \tag{25}$$

To determine the constant $M$, we use formula (18), as well as the relation

$$i(-\eta) = 2\int_0^1 \rho(\eta, \eta') i(\eta') \eta' \, d\eta', \tag{26}$$

which follows from (17) and (14). Multiplying (24) by $i(\eta)\eta$ and integrating with respect to $\eta$ from 0 to 1, with the aid of the indicated formulas we find

$$M = 2\int_{-1}^1 i^2(\eta) \eta \, d\eta. \tag{27}$$

Thus, for the desired brightness coefficients $\rho(\eta, \xi, \tau_0)$ and $\sigma(\eta, \xi, \tau_0)$ we have obtained the asymptotic formulas (22) and (23), in which the constants $M$ and $N$ are determined by formulas (25) and (27), and the function $u(\eta)$ is normalized according to (18).

To determine the brightness coefficients in the case of pure scattering (i.e., for $\lambda = 1$), in formulas (22) and (23) one must set $k \to 0$. For small $k$ we have

$$i(\eta) = 1 + \frac{3}{3 - x_1} k\eta, \tag{28}$$

where $x_1$ is the first coefficient in the expansion of the scattering indicatrix in Legendre polynomials. Substituting (28) into (27), we obtain

$$M = 8k/(3 - x_1), \tag{29}$$

and from formulas (25) and (18) it follows that

$$N = 1 - \frac{12k}{3 - x_1} \int_0^1 u(\eta)\eta^2 \, d\eta, \tag{30}$$

where the function $u(\eta)$ now refers to the case $\lambda = 1$.

Substitution of (29) and (30) into (22) and (23) for $k = 0$ gives

$$\rho(\eta, \xi, \tau_0) = \rho(\eta, \xi) - {}^4/_3 u(\eta)u(\xi)/[(1 - x_1/3)\tau_0 + \delta], \tag{31}$$

$$\sigma(\eta, \xi, \tau_0) = {}^4/_3 u(\eta)u(\xi)/[(1 - x_1/3)\tau_0 + \delta], \tag{32}$$

where

$$\delta = 2 \int_0^1 u(\eta)\eta^2 \, d\eta \Big/ \int_0^1 u(\eta)\eta \, d\eta. \tag{33}$$

Recently, in an article by van de Hulst and Grossman (6), formulas analogous to (22) and (23) were presented, but obtained by another method. They are a generalization of formulas previously obtained by the author (1, 7) for the case of isotropic scattering.

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