ON THE APPROXIMATION OF CONTINUOUS PERIODIC FUNCTIONS BY LINEAR OPERATORS CONSTRUCTED ON THE BASIS OF THEIR FOURIER SERIES

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 517.512

MATHEMATICS

M. F. TIMAN

ON THE APPROXIMATION OF CONTINUOUS PERIODIC FUNCTIONS BY LINEAR OPERATORS CONSTRUCTED ON THE BASIS OF THEIR FOURIER SERIES

(Presented by Academician S. N. Bernstein on 28 XII 1967)

Let $f(x)$ be a continuous periodic function of period 2π with Fourier series

$$\sum_{n=0}^{\infty} A_n(x),$$

where $A_0(x) = a_0/2, \quad A_n(x) = a_n \cos nx + b_n \sin nx \ (n = 1, 2, ...)$. With the aid of an arbitrary triangular matrix of numbers $\{\lambda_\nu(n)\}$ $(\lambda_0(n) = 1,\ \lambda_\nu(n) = 0$ for $\nu > n)$, we construct the sequence of operators

$$U_n(f; x; \lambda) = \sum_{\nu=0}^{n} \lambda_\nu(n) A_\nu(x) \quad (n = 0, 1, 2, ...). \tag{1}$$

Along with operators of the form (1), let us introduce for consideration the operators

$$U_r^*(f; x; \gamma) = \sum_{\nu=0}^{\infty} \gamma_\nu(r) S_\nu(f; x), \tag{2}$$

where

$$S_\nu(f; x) = \sum_{k=0}^{\nu} A_k(x),$$

and the system of functions $\{\gamma_\nu(r)\}$ is defined on some set E of the real axis and has the following properties: 1)

$$\sum_{\nu=0}^{\infty} \gamma_\nu(r) = 1$$

for every $r \in E$; 2) the series (2) converges uniformly for every $r \in E$.

Below we establish the following two assertions, giving estimates of the deviation of the function $f(x)$ from operators of the form (1) or (2) in terms of the sequence of its best approximations:

$$E_n(f) = \inf_{T_n} \max_x |f(x) - T_n(x)|.$$

Theorem 1. *For every continuous function $f(x)$ of period 2π, the inequality*

$$|f(x) - U_n(f;x;\lambda)| \leq (1 + L_n)E_n(f) + 2\sum_{\nu=0}^{m-1} \rho(2^{\nu+1};n)E_{2^\nu-1}(f) + \\ + 2\rho(n;n)E_{2^m}(f), \tag{3}$$

holds, where

$$2^m < n \leq 2^{m+1}, \quad L_n = \frac{2}{\pi}\int_0^\pi \left|\frac{1}{2} + \sum_{\nu=1}^n \lambda_\nu(n)\cos\nu x\right| dx, \quad \rho(\mu;n) =$$

$$= \frac{1}{\mu}\sum_{k=0}^{2\mu-1} |F_{k,\mu}(\lambda)|, \quad F_{k,\mu}(\lambda) = \frac{1-\lambda_\mu(n)}{2} + \sum_{\nu=0}^{\mu-1}(1 - \lambda_{\mu-\nu}(n))\cos\frac{\nu k\pi}{n}.$$

Theorem 2. *For the deviation of a continuous function $f(x)$ of period 2π from operators of the form (2), for every $r \in E$ the estimate*

$$|f(x) - U_r^*(f;x;\gamma)| \leq 2\gamma_0(r)E_0(f) + 12\sum_{\nu=0}^\infty E_{2^\nu}(f)\delta(\nu;r), \tag{4}$$

holds, where

$$\delta(\nu;r) = \left\{2^\nu \sum_{k=2^\nu}^{2^{\nu+1}-1} \gamma_k^2(r)\right\}^{1/2}.$$

We note that from the general inequalities (3) and (4) one can easily obtain, as consequences, the known (see ([1–3])) estimates for the deviation of continuous functions of period 2π from Fejér sums, Abel–Poisson sums, Zygmund means, etc. In particular, for Fejér sums, inequality (3) implies the estimate

$$\left|f(x) - \frac{1}{n+1}\sum_{\nu=0}^n S_\nu(f;x)\right| \leq \frac{10}{n+1}\sum_{\nu=0}^n E_\nu(f). \tag{5}$$

The method applied by S. B. Stechkin ([1]), using estimates of deviation for Vallée-Poussin sums, enabled him in this case to obtain the inequality

$$\left|f(x) - \frac{1}{n+1}\sum_{\nu=0}^n S_\nu(f;x)\right| \leq \frac{12}{n+1}\sum_{\nu=0}^n E_\nu(f).$$

Proof of Theorem 1. Let

$$T_\mu(x) = \sum_{\nu=0}^{\mu} C_\nu(x) = \sum_{\nu=0}^{\mu} \alpha_\nu \cos \nu x + \beta_\nu \sin \nu x$$

be an arbitrary trigonometric polynomial, and

$$U(T_\mu; x; \lambda) = \sum_{\nu=0}^{\mu} \lambda_\nu(n) C_\nu(x) \qquad (\mu \le n).$$

The identity holds

$$U(T_\mu; x; \lambda) = T_\mu(x) - \frac{1}{\mu} \sum_{k=0}^{2\mu-1} (-1)^k T_\mu \left(x + \frac{k\pi}{\mu} \right) F_{k,\mu}(\lambda), \tag{6}$$

where the numbers $F_{k,\mu}(\lambda)$ are defined in Theorem 1. Identity (6) is verified directly for the functions $\sin \nu x$, $\cos \nu x$ ($\nu = 0, 1, \dots, \mu$), after which its validity for the polynomial $T_\mu(x)$ follows. Now, choosing a sequence of polynomials $\{T_n(x)\}$ which, for each n, realize the best uniform approximation of order n to the function $f(x)$, we introduce into consideration the functions

$$R(\mu; \nu; n; x) = \frac{1}{\mu} \sum_{k=0}^{2\mu-1} (-1)^k T_\mu \left(x + \frac{k\pi}{\mu} \right) F_{k,\nu}(\lambda),$$

where $1 \le \mu \le \nu \le n$. It is not difficult to verify that these functions satisfy the following relations: for $\nu > \mu$, $R(\mu; \nu; n; x) = R(\mu; \mu; n; x)$, and $R(n; n; n; x) = T_n(x) - U(T_n; x; \lambda)$. The estimate is known (see ([3]), p. 591)

$$|f(x) - U_n(f; x; \lambda)| \le (1 + L_n) E_n(f) + \max_x |T_n(x) - U(T_n; x; \lambda)|. \tag{7}$$

Using the above-indicated properties of the functions $R(\mu; \nu; n; x)$, we find that for $2^m < n \le 2^{m+1}$,

$$\begin{aligned} |T_n(x) - U(T_n; x; \lambda)| \le |R(2; 2; n; x) - R(0; 2; n; x)| + |R(n; n; n; x) - R(2^m; n; n; x)| \\ + \sum_{\nu=1}^{m-1} |R(2^{\nu+1}; 2^{\nu+1}; n; x) - R(2^\nu; 2^{\nu+1}; n; x)| \end{aligned}$$

$$\big(R(0; \nu; n; x) = T_0(x) - U(T_0; x; \lambda)\big). \tag{8}$$

Applying identity (6) to the polynomial $P_{2^{\nu+1}}(x) = T_{2^{\nu+1}}(x) - T_{2^\nu}(x)$ $(0 \le \nu \le m-1)$, we obtain

$$R(2^{\nu+1}; 2^{\nu+1}; n; x) - R(2^\nu; 2^{\nu+1}; n; x) =$$

$$= \frac{1}{2^{\nu+1}} \sum_{k=0}^{2^{\nu+2}-1} (-1)^k P_{2^{\nu+1}} \left(x + \frac{k\pi}{2^{\nu+2}} \right) F_{k,2^{\nu+1}}(\lambda).$$

Hence it follows that

$$\left| R\left(2^{\nu+1}, 2^{\nu+1}, n; x\right) - R\left(2^\nu, 2^{\nu+1}, n; x\right) \right| \leqslant 2E_{2^\nu}(f)\rho\left(2^{\nu+1}, n\right). \tag{9}$$

In view of (8) and (9), we find that

$$|T_n(x) - U(T_n; x; \lambda)| \leqslant 2 \sum_{\nu=0}^{m-1} E_{2^\nu - 1}(f)\rho\left(2^{\nu+1}, n\right) + E_{2^m}(f)\rho(n; n). \tag{10}$$

From the estimates (7) and (10) inequality (3) follows.

Theorem 2 is a consequence of the following stronger assertion.

Theorem 3. Let $f(x)$ be a continuous function of period 2π, and let $\{\gamma_\nu(r)\}$ be an arbitrary sequence of functions $(\nu = 0, 1, 2, ...)$ defined on some set E of the real axis. Then from the convergence of the series

$$\sum_{\nu=0}^{\infty} E_{2^\nu}(f)\delta(\nu; r) \qquad (r \in E)$$

it follows that, for any x, the series

$$\sum_{\nu=0}^{\infty} |\gamma_\nu(r)| \, |f(x) - S_\nu(f; x)|$$

converges and, moreover, the estimate

$$\sum_{\nu=0}^{\infty} |\gamma_\nu(r)| \, |f(x) - S_\nu(f; x)| \leqslant 2E_0(f)\gamma_0(r) + 12 \sum_{\nu=0}^{\infty} E_{2^\nu}(f)\delta(\nu; r), \tag{11}$$

holds, where $\delta(\nu; r)$ are the numbers defined in Theorem 2.

Proof. Let $\{T_n(x)\}$ be a sequence of trigonometric polynomials which, for each n ($n = 0, 1, 2, ...$), give the best approximation of order n to the function $f(x)$ in the uniform metric. Obviously, for any $m = 0, 1, 2, ...$

$$\theta_m(f;x;r) = \sum_{\nu=2^m}^{2^{m+1}-1} |\gamma_\nu(r)|\,|f(x) - S_\nu(f;x)| =$$

$$= \sum_{\nu=2^m}^{2^{m+1}-1} |\gamma_\nu(r)| \left| \frac{1}{\pi} \int_0^\pi \varphi_m(x;t) \frac{\sin(\nu+1/2)t}{2\sin t/2}\, dt \right|,$$

where

$$\varphi_m(x;t) = f(x+t) - T_{2^m}(x+t) - 2[f(x) - T_{2^m}(x)] + f(x-t) - T_{2^m}(x-t).$$

Since $|\varphi_m(x;t)| \leqslant 4E_{2^m}(f)$, we find

$$I_1 = \left| \frac{1}{\pi} \int_0^{\pi/2^m} \varphi_m(x;t) \frac{\sin(\nu+1/2)t}{2\sin t/2}\, dt \right| \leqslant 2(2\nu+1)\frac{1}{2^m} E_{2^m}(f). \tag{12}$$

Let us now estimate the integral

$$I_2 = \left| \frac{1}{\pi} \int_{\pi/2^m}^{\pi} \frac{\varphi_m(x;t)}{2\,\mathrm{tg}\,t/2} \sin\nu t\, dt + \frac{1}{2\pi} \int_{\pi/2^m}^{\pi} \varphi_m(x;t) \cos\nu t\, dt \right| \leqslant$$

$$\leqslant 2E_{2^m}(f) + \left| \frac{1}{\pi} \int_{\pi/2^m}^{\pi} \frac{\varphi_m(x;t)}{2\,\mathrm{tg}\,t/2} \sin\nu t\, dt \right|. \tag{13}$$

Fixing x, consider the function

$$\psi_m(x;t) = \begin{cases} \dfrac{\varphi_m(x;t)}{4\,\mathrm{tg}\,t/2}, & \dfrac{\pi}{2^m} \leqslant t \leqslant \pi, \\ 0, & 0 \leqslant t < \dfrac{\pi}{2^m}, \end{cases}$$

$$\psi_m(x;-t) = -\psi_m(x;t), \qquad \psi_m(x;t+2\pi) = \psi_m(x;t).$$

The second term on the right-hand side of inequality (13), for any $\nu = 1, 2, ...$, is a Fourier coefficient of the function $\psi_m(x;t)$. Therefore, by Bessel' s inequality,

$$\left\{\sum_{\nu=2^m}^{2^{m+1}-1}\left|\frac{1}{\pi}\int_{\pi/2^m}^{\pi}\frac{\varphi_m(x;t)}{2\operatorname{tg}t/2}\sin\nu t\,dt\right|^2\right\}^{1/2}\leq\left\{\frac{2}{\pi}\int_0^{\pi}|\psi_m(x;t)|^2\,dt\right\}^{1/2}\leq\frac{2}{\sqrt{\pi}}2^{m/2}E_{2^m}(f). \tag{14}$$

Using estimates (12), (13), (14) and applying Bunyakovsky' s inequality, we obtain

$$\theta_m(f;x;r)\leq\frac{2E_{2^m}(f)}{2^m}\sum_{\nu=2^m}^{2^{m+1}-1}|\gamma_\nu(r)|(2\nu+1)+2E_{2^m}(f)\sum_{\nu=2^m}^{2^{m+1}-1}|\gamma_\nu(r)|$$
$$+\sum_{\nu=2^m}^{2^{m+1}-1}|\gamma_\nu(r)|\left|\frac{1}{\pi}\int_{\pi/2^m}^{\pi}\frac{\varphi_m(x;t)}{2\operatorname{tg}t/2}\sin\nu t\,dt\right|\leq 12E_{2^m}(f)\delta(m;r).$$

From this estimate it follows that

$$\sum_{m=0}^{\infty}\theta_m(f;x;r)\leq 12\sum_{m=0}^{\infty}E_{2^m}(f)\delta(m;r).$$

In an analogous way the following assertion is also established.

Theorem 4. If $\{\gamma_\nu(n)\}$ is an arbitrary triangular matrix of numbers, i.e. $\gamma_\nu(n)=0$ $(\nu>n)$, then

$$\sum_{\nu=0}^{n}|\gamma_\nu(n)|\,|f(x)-S_\nu(f;x)|\leq 12\left\{\sum_{\nu=0}^{m-1}E_{2^\nu-1}(f)\delta(\nu;n)+E_{2^m}(f)\left(2^m\sum_{\nu=2^m}^{n}\gamma_\nu^2(n)\right)^{1/2}\right\},$$

where

$$\delta(\nu;n)=\left\{2^\nu\sum_{k=2^\nu}^{2^{\nu+1}-1}\gamma_k^2(n)\right\}^{1/2},\qquad 2^m<n\leq 2^{m+1}.$$

From Theorems 3 and 4 there follows a series of corollaries for classical summability methods. We give one of them.

Corollary. If $\gamma_\nu(n)=1/(n+1)$, $\nu=0,1,\ldots,n$, $\gamma_\nu(n)=0$ for $\nu>n$, then

$$\frac{1}{n+1}\sum_{\nu=0}^{n}|f(x)-S_\nu(f;x)|\leq\frac{C}{n+1}\sum_{\nu=0}^{n}E_\nu(f). \tag{15}$$

We note that in the case when $E_n(f) = (n+1)^{-\alpha}$, $(0 < \alpha \leq 1)$, Aleksich and Kralik ([4]) obtained the estimate

$$\frac{1}{n+1}\sum_{\nu=0}^{n}|f(x) - S_\nu(f;x)| \leq C \begin{cases}(n+1)^{-\alpha}, & 0 < \alpha < 1,\\ (n+1)^{-1}\ln(n+1), & \alpha = 1,\end{cases}$$

which follows from inequality (15).

The results of the present note, with proofs, were presented by the author at the scientific school on summability theory (Sverdlovsk, 12 VII 1967).

Dnepropetrovsk
Agricultural Institute

Received
25 XII 1967

CITED LITERATURE

1. S. B. Stechkin, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, **62**, 48 (1961).

2. M. F. Timan, *DAN*, **145**, No. 4, 741 (1962).

3. M. F. Timan, *Izv. AN SSSR, ser. matem.*, **29**, 3, 587 (1965).

4. G. Alexitz, D. Kralik, *Acta math.*, **16**, F. 1–2, 43 (1965).

Note: Figure translations are in progress. See original paper for figures.