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Abstract

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MATHEMATICS

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ON THE PROBLEM OF MODULI OF RIEMANN SURFACES

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1. The classical problem of moduli of Riemann surfaces has as its starting point B. Riemann' s remark ([13]) that the classes of conformally equivalent closed surfaces of genus $g > 1$ depend on $3g-3$ complex parameters, called **moduli**, and consists in studying the nature of these parameters and, if possible, introducing these parameters so that they define a complex-analytic structure in the corresponding space of Riemann surfaces. In investigations of this problem by various authors it became clear that conformal equivalence of Riemann surfaces should be replaced by a weaker notion of equivalence, in which topological restrictions are imposed on the conformal mappings.

A **surface of type** (g, n) is a surface S obtained from a closed Riemann surface of genus g by deleting $n \geqslant 0$ distinct points. In what follows it is assumed everywhere that $3g - 3 + n > 0$. We shall call a Riemann surface S of type (g, n), together with its canonical dissection fixed up to a homeomorphism of S onto itself homotopic to the identity, a **marked** Riemann surface. The consideration of homeomorphisms of one marked surface onto another is equivalent to specifying the homotopy class of homeomorphisms of the corresponding Riemann surfaces. The **Teichmüller space** $T_{g,n}$ is the set of classes $[S]$ of conformally equivalent marked surfaces S of type (g, n), with metric

$$\rho([S], [S']) = \inf \log K(f),$$

where the infimum is taken over all K-quasiconformal homeomorphisms $f : S \to S'$ with $K = K(f) < \infty$, and S and S' are concrete representatives of the classes $[S]$ and $[S']$. For $n = 0$ the surfaces under consideration are closed.

R. Fricke (see ([15])) and O. Teichmüller (([14]), see also ([2,4,10])) proved that the space $T_{g,n}$ is homeomorphic to the $(6g - 6 + 2n)$-dimensional Euclidean space $E^{6g-6+2n}$. In the works ([3,5,7,9,12]) it was proved that a complex-analytic structure can be introduced in $T_{g,n}$. This structure turns $T_{g,n}$ into a complex-analytic manifold of dimension $3g-3+n$ and determines the coordinates of the points of the space $T_{g,n}$, i.e. the moduli of the corresponding marked surfaces, only locally. L. Ahlfors ([1]) and L. Bers ([6]) proved that there exists a biholomorphic homeomorphism of the space $T_{g,n}$ onto a bounded domain $B_m \subset C^m$, $m = 3g-3+n$. The coordinates of the points of the domain B_m are global moduli of the points of the space $T_{g,n}$. Below some elements of the construction ([1,6]) will be used. We also note the work ([8]), in which a global real analytic structure in $T_{g,n}$ is defined.

2. In the present paper new moduli of marked Riemann surfaces are introduced, having a more transparent character. These moduli are equivalent to the global moduli of Ahlfors–Bers and connect the problem of moduli of Riemann surfaces with the problem of coefficients for a certain class of univalent analytic functions.

Let U_1 be the disk $|z| < 1$, U_2 the domain $1 < |z| \leq \infty$, and let Γ be a Fuchsian group of the first kind without elliptic elements, generated by a finite number of generators, for which the circle $\gamma : |z| = 1$ is a limiting circle. Then, as is known, $S = U_1/\Gamma$ is a surface of type (g, n), where $m = 3g-3+n > 0$, and the surface $\overline{S} = U_2/\Gamma$ is the mirror image of the surface S. Denote by $B(U_1, \Gamma)$ the complex Banach space of functions $\varphi(z)$ analytic in the disk U_1 and satisfying the condition $\varphi(Az){A'}^2(z) = \varphi(z)$ for all $A \in \Gamma$, with norm

$$\|\varphi\|_{B(U_1,\Gamma)} = \sup_{|z|\leq 1}(1-|z|^2)^2|\varphi(z)| < \infty.$$

By the Riemann-Roch theorem, $\dim B(U_1, \Gamma) = m$.

Lemma. Let $\varphi_1(z), \varphi_2(z), \dots, \varphi_m(z)$ be a basis of the space $B(U_1, \Gamma)$. Then the Wronskian determinant

$$W_m(z) = \det \|\varphi_k^{(p)}(z)\|, \qquad k = 1, 2, \dots, m; \quad p = 0, 1, \dots, m-1,$$

on any compact set $F \subset U_1$ can vanish only at a finite number of points.

Let $\varphi_1(z), \varphi_2(z), \dots, \varphi_m(z)$ be a basis of the space $B(U_1, \Gamma)$ fixed for what follows. If $W_m(0) = 0$, then we pass to the conjugate group $\Gamma_1 = C_0 \Gamma C_0^{-1}$, where C_0 is a fractional-linear mapping of the disk U_1 onto itself that carries the point z_0, at which $W_m(z_0) \neq 0$, to zero. Therefore one may assume that $W_m(0) \neq 0$.

Let $B_1(\Gamma)$ be the set of complex-valued functions $\mu(z)$ measurable in the z-plane such that $\mu(z)d\overline{z}/dz$ is invariant with respect to Γ, $\|\mu\|_{L_\infty(U_2)} < 1$, and $\mu(z) = 0$ for $z \in U_1$. For each $\mu \in B_1(\Gamma)$ there exists a unique quasiconformal homeomorphism $w = f_\mu(z)$ of the z-plane onto itself, satisfying the Beltrami equation $w_{\overline{z}} = \mu w_z$ and normalized by the conditions $f_\mu(0) = 0$, $f'_\mu(0) = 1$, $f_\mu(\infty) = \infty$. Then $\Gamma^\mu = w_\mu \Gamma (w_\mu)^{-1}$ is a quasifuchsian group of fractional-linear mappings of the w-plane with invariant curve $\gamma^\mu = f_\mu(\gamma)$, isomorphic to the group Γ, and $S^\mu = f_\mu(U_2)/\Gamma^\mu$ is a Riemann surface. When μ runs through the set $B_1(\Gamma)$, we obtain any point $[S^\mu]$, $S^\mu = f_\mu(\overline{S})$, of the Teichmüller space

$T(S) = T_{g,n}$. We shall call the functions μ and ν **equivalent** if they determine the same point of the space $T_{g,n}$, i.e., if the surfaces S^μ and S^ν are conformally equivalent and the mappings f_μ and f_ν are homotopic. The equivalence of the functions μ and ν from $B_1(\Gamma)$ is equivalent to the fact that $f_\mu(z) = f_\nu(z)$ for $z \in U_1$. The corresponding equivalence class $[\mu]$ of functions $\mu \in B_2(\Gamma)$, the class of mappings $w_\mu(z)$, will be denoted by $[w_\mu]$.

Theorem 1. Let $w = f_\mu(z) \in [w_\mu]$ and

$$f_\mu(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad z \in U_1.$$

If the function

$$g_\nu(z) = z + \sum_{k=2}^{\infty} b^k, \qquad z \in U_1$$

is the restriction, for $z \in U_1$, of a homeomorphism $w = f_\nu(z)$ from some class $[w_\nu]$ and satisfies the conditions $b_k = a_k$, $k = 2, 3, \dots, m+1$, then $f_\nu(z) = f_\mu(z)$ in U_1 and $[w_\mu] = [w_\nu]$.

For the proof one considers the Schwarzian derivative $\varphi^\mu(z) = \{f_\mu, z\}$ of the homeomorphism $f_\mu(z)$ for $|z| < 1$. As is known ([11]), $\varphi^\mu \in B(U_1, \Gamma)$. Substituting into the equality

$$\varphi^\mu(z) = \sum_{l=0}^{m} \omega_l \varphi_l(z), \qquad \omega_l = \text{const},$$

the Taylor expansions

$$\varphi^\mu(z) = \sum_{k=0}^{\infty} \alpha_k z^k \quad \text{and} \quad \varphi_l(z) = \sum_{k=0}^{\infty} \alpha_{kl} z^k, \quad |z| < 1,$$

and equating coefficients of like powers z^k, $k = 0, 1, \dots, m-1$, we obtain, for determining $\omega_1, \dots, \omega_m$, an algebraic system

of the equations

$$\sum_{l=1}^{m} a_{kl} \omega_l = a_k, \qquad k = 0, 1, \dots, m-1,$$

whose determinant $\det \|a_{kl}\|$, together with $W_m(0)$, is different from zero. Hence, noting that the coefficients $a_0, a_1, \dots, a_{m-1}$ are uniquely expressed in terms of $a_2, a_3, \dots, a_{m+1}$ by known formulas, we obtain that $\varphi^\mu(z)$ is uniquely determined by the coefficients $a_2, \dots, a_{m+1}$. Consequently, $\varphi^\mu(z) = \varphi^\nu(z)$ for $z \in U_1$, and then, by what was proved in ([1]), we have $f_\mu(z) = f_\nu(z)$ for $z \in \overline{U}_1$, and therefore $[w_\mu] = [w_\nu]$.

The point with coordinates $(a_2, a_3, \dots, a_{m+1})$—the coefficients of the homeomorphisms $f_\mu(z)$ from the fixed class $[w_\mu]$—will be denoted by a_μ.

Theorem 2. *The mapping*

$$\tau : [\mu] \to a_\mu = (a_2, a_3, \dots, a_{m+1})$$

is a biholomorphic homeomorphism of the space $T_{g,n} = T(S)$ onto a bounded domain $D_m \subset C^m$.

(Holomorphy is understood here in the sense that if $\mu \in B_1(\Gamma)$, as an element of $L_\infty(U_2)$, depends holomorphically on complex parameters, then $a_2, a_3, \dots, a_{m+1}$ also depend holomorphically on these parameters, and conversely.*) The continuity and holomorphy of the mapping $\tau\mu = a_\mu$ is established with the aid of the variational formula

$$w_\lambda(z) = z - \frac{z^2}{\pi} \iint_{U_2} \frac{\lambda(\zeta)\, d\sigma(\zeta)}{\zeta^2(\zeta - z)} + O(\varepsilon^2), \qquad |z| \le R < \infty,$$

where $\varepsilon = \|\lambda\|_{L_\infty(U_2)}$ is small; this formula is applied to the mapping $w_{\mu+\nu}(w_\mu)$, when $\|\nu\|_{L_\infty(U_2)}$ is small. Hence, using the result mentioned above ([1,6]) that the mapping $\varkappa : [\mu] \to \varphi^\mu(z)$ is a biholomorphic homeomorphism of $T_{g,n}$ onto a domain $B_m \subset C^m$, by Theorem 1 we obtain that the mapping

$$\tau \circ \varkappa^{-1} : B_m \to D_m$$

(and together with it also τ) is a biholomorphic homeomorphism and D_m is a domain in C^m.

By Theorems 1 and 2, the coordinates $(a_2, a_3, \dots, a_{m+1})$ of the points of the domain D_m are global moduli of the points of the space $T_{g,n}$. All known properties of the domain B_m (see ([6])), invariant under biholomorphic mappings (for example, holomorphic convexity, homogeneity), are transferred also to the domain D_m.

It follows from Theorems 1 and 2 that the moduli problem for Riemann surfaces is equivalent to a certain coefficient problem for the class of univalent functions under consideration $f_\mu(z)$ for $|z| < 1$, i.e. to the problem of finding the range of values of $(a_2, a_3, \dots, a_{m+1})$; and, since g and n may take arbitrary natural values, obtaining quantitative estimates for the boundary points of D_m is also connected with the solution of the coefficient problem in the class of all functions univalent in the disk $|z| < 1$,

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

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* More precisely, for given $a_2, a_3, \ldots, a_{m+1}$ there is found a certain function $\mu \in B_1(\Gamma)$, depending holomorphically on them.

Note: Figure translations are in progress. See original paper for figures.