TRANSVERSAL MAPPINGS OF FOLIATIONS

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Abstract

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1. Introduction. A distribution (see (1)) on a smooth manifold M (which in what follows will be assumed connected, without boundary, but not necessarily compact), defined by a subbundle ξ of the tangent bundle $\tau(M)$, will be denoted by (M, ξ). The dimension $\dim T$ of the distribution $T = (M, \xi)$ will mean the dimension $\dim \xi$, and its codimension $\operatorname{codim} T$ the difference $\dim M - \dim \xi$. Involutive distributions (see (1)) will be called foliations. If two distributions $S = (M, \xi)$, $T = (N, \theta)$ and a smooth mapping $f : M \to N$ are given, then by $D_f : \xi \to \tau(N)/\theta$ we denote the composition of three homomorphisms: the inclusion $\xi \to \tau(M)$, the differential $d_f : \tau(M) \to \tau(N)$, and the projection $\tau(N) \to \tau(N)/\theta$. We shall call the mapping f transversal with respect to the distributions S, T if the homomorphism D_f is injective (i.e., its restriction to each fiber is injective).

On every manifold M there exist two standard distributions $T_0(M) = (M, 0)$, where 0 is the zero-dimensional bundle, and $T_\tau(M) = (M, \tau)$, where τ is the tangent bundle $\tau(M)$. It is obvious that a smooth mapping $f : M \to N$ is an embedding (immersion) if and only if this mapping is transversal with respect to the distributions $T_\tau(M), T_0(N)$.

If $S = (M, \xi)$ and $T = (N, \theta)$ are distributions, then by $i(S, T)$ we denote the space of all smooth mappings $M \to N$ that are transversal with respect to S, T. If $K \subset M$ is an arbitrary submanifold, then by $I_K(\xi, \tau(N)/\theta)$ we denote the space of all injections of the restriction ξ_K of the bundle ξ to K into the bundle $\tau(N)/\theta$. The correspondence $f \to D_f$ defines a continuous mapping

$$D : i(S, T) \to I_M(\xi, \tau(N)/\theta).$$

It was shown by S. Smale and M. Hirsch in (6, 4) that, for $\dim M < \dim N$, the mapping

$$D_* : \pi_0(i(T_\tau(M), T_0(N))) \to \pi_0(I_M(\tau(M), \tau(N)))$$

is one-to-one and onto. In fact, in these works it is proved, though not explicitly formulated, that for $\dim M < \dim N$ the mapping D is a weak homotopy equivalence. In the present paper this result is extended to the case of more general distributions.

Theorem 1. *Let a foliation $S = (M, \xi)$ and a distribution $T = (N, \theta)$ be given, with $\dim S \leq \operatorname{codim} T$. Then the mapping*

$$D : i(S, T) \to I_M(\xi, \tau(N)/\theta)$$

is a weak homotopy equivalence.

The theorem 2, corollaries A, B, C, and theorem 3 given below follow directly from theorem 1. If $S = (M, \xi)$ is a foliation, then a smooth mapping $f : M \to N$ will be called an S-embedding if the restriction of f to the leaves of the foliation S is an embedding, or, equivalently, if the mapping f is transversal with respect to the distributions $S, T_0(N)$.

Theorem 2. *If $S = (M, \xi)$ is a foliation, then for the existence of an S-embedding $f : M \to N$ homotopic to a given continuous mapping $g : M \to N$, with $\dim \xi < \dim N$, it is necessary and sufficient that there exist a vector bundle α on the manifold M such that*

$$\alpha \oplus \xi = g!(\tau(N)).$$

Corollary A. *Suppose that on the manifold M a smooth action without ne-*

fixed points is the group R^1. Then there exists a smooth mapping $f : M \to R^2$, whose restriction to each trajectory is an immersion.

Corollary B. For any k-dimensional foliation S on Euclidean space R^n there exists a smooth S-immersion $f : R^n \to R^{k+1}$.

Corollary C. If on a manifold M there exist q linearly independent commuting vector fields, then there exists a smooth mapping $f : M \to R^{q+1}$, the rank of whose differential at every point is not less than q.

Let $T = (M, \xi)$ be a distribution on a compact oriented n-dimensional manifold M. We shall call a homology class $x \in H_i(M, Z)$ transversally realizable with respect to the distribution T if there exist a smooth closed oriented i-dimensional manifold X and a smooth mapping $f : X \to M$, transverse with respect to the distributions $T_\tau(X), T$, such that the image of the fundamental class of the manifold X is carried by the mapping f into the class x.

If α is a vector bundle over the manifold M, then denote by x_α the cohomology class from the group $H^{n-i+\dim \alpha}(M^\alpha, Z)$ (where M^α is the Thom space of the bundle α) obtained from the class x by applying Poincaré duality and the Thom isomorphism.

Theorem 3. In order that the class x be transversally realizable with respect to the distribution T for $n - i > k = \dim \xi$, it is necessary and sufficient that, for a vector bundle α of sufficiently high dimension such that $\alpha \oplus \xi = E$ (E is the trivial bundle), the cohomology class x^{α} be realizable with respect to the group $SO(n-k-i)$ (see (7)).

Corollary. For any $x \in H_i(M, Z)$ with $k < n-i$ there exists a natural number m such that the class mx is transversally realizable with respect to T.

We note that a result analogous to Theorem 3 is valid for homology classes modulo 2 of nonorientable manifolds, and also for homology classes with noncompact carriers in the case of open manifolds.

Plan of the further exposition. Theorem 1 is a direct consequence of Theorems 4 and 6. The proof of Theorem 5 and of Theorem 6 following from it rests on Propositions 1 and 2. In addition to Theorems 5 and 6, in § 4 some corollaries of Theorem 6 are formulated which supplement Theorem 1. In § 5 results of a geometric character are given, whose proof can be carried out according to the same scheme as the proof of Theorem 1.

2. Triangulation of foliations

For each pair of nonnegative integers k, p, where $k \geq p$, fix a Euclidean space R^k and a rectilinear p-dimensional simplex $s_p \subset R^k$. The direct product $s_p \times s_q \subset R^k \times R^l$ will be called the standard bicell s^{kl}_{pq}.

Consider on a manifold M a foliation T, and put $k = \dim T$, $l = \operatorname{codim} T$. A cellular decomposition $\widetilde{M}$ of the manifold M^n will be called regular with respect to the foliation T if for every closed cell $\sigma \in \widetilde{M}$ there exist nonnegative numbers p, q and a diffeomorphism f_σ of some neighborhood $U \subset R^k \times R^l$ of the standard bicell $s^{kl}_{pq} \subset R^k \times R^l$ into the manifold M^n, possessing two properties: a) the diffeomorphism f_σ carries the k-dimensional layers into which the neighborhood $U \subset R^k \times R^l$ is naturally decomposed into layers of the foliation T; b) the bicell $s^{kl}_{pq} \subset U$ is thereby mapped diffeomorphically onto the cell σ. We note that a cellular decomposition regular with respect to T is a subdivision of some smooth triangulation of the manifold M^n. Whitehead' s triangulation theory ([8]) can be transferred to smooth foliations; in particular, the following is valid.

Theorem 4. For any foliation T on a smooth manifold M there exists a cellular decomposition $\widetilde{M}$ regular with respect to T.

3. **Homotopy sheaves.** With each CW-complex K there is associated the category $\mathcal{K}$ of all its (closed) subcomplexes and their inclusions into one another. A contravariant functor from $\mathcal{K}$ to the category $\mathcal{T}$ of topological spaces and continuous maps will be called a topological presheaf over K. Since the closed subcomplexes of the complex K satisfy the axioms for the open sets of a certain topology, and the objects of the category $\mathcal{T}$ are sets while the morphisms are maps, one can (see ([2])) single out the notion of a topological sheaf over K. A presheaf (sheaf) will be called a homotopy presheaf (sheaf) if its values on the morphisms of the category $\mathcal{K}$ are Serre fibrations. We shall call a homomorphism $\Phi_1 \to \Phi_2$ of presheaves Φ_1, Φ_2 over the complex K a natural transformation of functors $\Phi_1 \to \Phi_2$. A

homomorphism $\varphi : \Phi_1 \to \Phi_2$ will be called a weak homotopy equivalence if, for any subcomplex $L \subset K$, the continuous map $\varphi_L : \Phi_1(L) \to \Phi_2(L)$ is a weak homotopy equivalence.

Proposition 1. *In order that a topological sheaf Φ over the complex K be a homotopy sheaf, it is necessary and sufficient that, for every closed cell $\bar{\sigma} \subset K$ and inclusion $i : \dot{\sigma} \to \bar{\sigma}$ of the boundary $\dot{\sigma}$, the continuous map $\Phi(i)$ be a Serre fibration.*

Proposition 2. *In order that a homomorphism $\varphi : \Phi_1 \to \Phi_2$ of homotopy sheaves Φ_1, Φ_2 over the complex K be a weak homotopy equivalence, it is necessary and sufficient that, for every closed cell $\bar{\sigma} \subset K$, the continuous map $\varphi_{\bar{\sigma}} : \Phi_1(\bar{\sigma}) \to \Phi_2(\bar{\sigma})$ be a weak homotopy equivalence.*

4. **Topological sheaves of germs of maps.** Let $S = (M, \xi)$ be a foliation, $T = (N, \theta)$ a distribution, $\widetilde{M}$ a decomposition of the manifold M regular with respect to S, and K a subcomplex of the decomposition $\widetilde{M}$.

Denote by $F_K(S, T)$ the topological sheaf over the complex K which assigns to each subcomplex $L \subset K$ the inductive limit of the spaces $i(S_{U_j}, T)$ over all neighborhoods $U_j \subset M$ of the complex $L \subset M$ (S_{U_j} is the restriction of the foliation S to the neighborhood U_j). To an inclusion $Q \subset L$ the sheaf F_K assigns the map arising, under passage to the inductive limit, from the restriction maps $I(S_U, T) \to I(S_V, T)$ $(U \supset V)$.

Theorem 5. *If $\dim S \leq \operatorname{codim} T$ and $\dim K < \operatorname{codim} T$, then the sheaf $F_K(S, T)$ is a homotopy sheaf.*

The proof of Theorem 5 in the special case where the complex K is a standard plaque is carried out by direct geometric arguments. The general case reduces to this special one by virtue of Proposition 1.

Denote by $\Phi_K(\xi, \tau(N)/\theta)$ the topological sheaf which assigns to each subcomplex $L \subset K$ the space $I_L(\xi, \tau(N)/\theta)$, and to an inclusion $Q \subset L$ the restriction $I_L(\xi, \tau(N)/\theta) \to I_Q(\xi, \tau(N)/\theta)$. It is clear that the sheaf $\Phi_K(\xi, \tau(N)/\theta)$ is a homotopy sheaf. For all open sets $U \subset M$ continuous maps $D : i(U_S, T) \to I_U(\xi, \tau(N)/\theta)$ are defined, by means of which a homomorphism $D_K : F_K(S, T) \to \Phi_K(\xi, \tau(N)/\theta)$ is constructed.

Theorem 6. *If $\dim S \leq \operatorname{codim} T$ and $\dim K < \operatorname{codim} T$, then D_K is a weak homotopy equivalence.*

The proof of Theorem 6 consists in reducing it, by means of Theorem 5 and Proposition 2, to the special case of a standard plaque. The proof in that case is carried out by simple geometric arguments.

An immediate consequence of Theorem 6 is

Theorem 7. *If $\dim S < \operatorname{codim} T$, then a continuous map $f : M \to N$, homotopic to a map $g : M \to N$ transverse with respect to S, T, can be approximated* *by a map F transverse with respect to S, T.*

Using the results of paper (5) and Theorem 6, we obtain:

Theorem 8. *If M is an open manifold, then the continuous mapping*

$$D: \ i(T_\tau(M), T) \to I_M(\tau(M), \tau(N)/\theta)$$

is a weak homotopy equivalence. (Here it is not assumed that $\dim M < \operatorname{codim} T$*.)*

The notion of transversality introduced by us generalizes to the case of foliations on continuous mappings.

Let S, T be foliations on manifolds M and N. A mapping $f: M \to N$ is called topologically transversal with respect to S, T if, for any two leaves $s \subset M$ and $t \subset N$ and any two points $\mu \in s$ and $\nu \in t$, there exist neighborhoods $U_\mu \subset s$ and $U_\nu \subset t$ such that the intersection $f^{-1}(U_\nu) \cap U_\mu$ consists of no more than one point. Using the argument of paper (3) and Theorems 6, 7, we obtain:

Theorem 9. *If $\frac{1}{2} \dim M < \operatorname{codim} T - \dim S$, then any continuous mapping $f: M \to N$, topologically transversal with respect to S, T, can be approximated by a smooth mapping transversal with respect to S, T.*

5. Metric theorems. Let distributions $S = (M, \xi)$, $T = (N, \theta)$ be given, and let a Riemannian metric be introduced in the manifold N. Denote by $i_\varepsilon(S, T) \subset i(S, T)$ the subset formed by those smooth mappings $f \in i(S, T)$ that have the following property: if $x_\nu \in \theta$ and $y_\nu \in \operatorname{Im} d_f$ are unit tangent vectors at the point $\nu \in N$, then $\langle x_\nu, y_\nu \rangle < \varepsilon$.

Theorem 10. *If S is a foliation,* $\dim S < \operatorname{codim} T$*, and* $0 < \varepsilon < 1$*, then the inclusion $i_\varepsilon(S, T) \subset i(S, T)$ is a weak homotopy equivalence.*

Theorem 1 of the present paper can be interpreted as a theorem on the existence of a secant surface of a special kind in the trivial skew product $M \times N \to M$. For an arbitrary smooth skew product $X \to M$, foliation (M, ξ), and distribution (X, θ), one can prove an analogous theorem. We formulate the corresponding result in one special case.

Theorem 11. *Let M be a manifold with affine connection and N a submanifold of smaller dimension. Then there exists on the manifold M a vector field X, affinely nondegenerate at each point $\nu \in N$. (For any vector $y_\nu \neq 0$, the covariant derivative $\nabla_{y_\nu}(x)$ is nonzero.)*

We note that in the case of a flat connection (i.e., a connection with trivial holonomy group), Theorem 11 is equivalent to Hirsch' s theorem on immersing a manifold in Euclidean space.

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Note: Figure translations are in progress. See original paper for figures.