EXTENSION OF K. ITÔ' S THEOREM

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Abstract

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MATHEMATICS

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EXTENSION OF K. ITÔ' S THEOREM

"ON THE INDEPENDENCE OF THE JUMPS OF A PROCESS"

TO PROCESSES WITH INDEPENDENT INCREMENTS DEFINED

ON TOPOLOGICAL GROUPS WITH A COUNTABLE BASE

(Presented by Academician A. N. Kolmogorov, January 9, 1968)

1. Let $\xi_t(\omega)$, $t \in [a, b]$, be a real, separable, stochastically continuous process with independent increments. Then it is known (1) that almost everywhere the trajectories $\xi_t(\omega)$ have no discontinuities of the second kind. Passing to a stochastically equivalent process, one can ensure that this process is, with probability 1, right-continuous. Let $\mathcal{A}$ be a system of Borel sets on the line lying outside some neighborhood of zero. Then the function $\nu(t, A; \omega)$, equal to the number of jumps $\xi_\tau(\omega) - \xi_{\tau-0}(\omega)$, $a \leq \tau \leq t$, into the set $A \in \mathcal{A}$, will be a random variable on the σ-algebra generated by the process $\xi_t(\omega)$.

If the sets $A_1, A_2, \dots, A_k$ in $\mathcal{A}$ do not intersect, then K. Itô' s theorem asserts that the random variables $\nu(t, A_1; \omega), \dots, \nu(t, A_k; \omega)$ are independent. For sets from $\mathcal{A}$, the quantity $\nu(t, A; \omega)$ will be a measure. Such measures are called measures with independent values.

2. The definitions of random variables taking values in groups and their simplest properties may be found, for example, in the survey article (2). All measures occurring in this note are assumed to be Borel measures.

Let $\xi_t(\omega)$, $t \in [a, b]$, be a stochastically continuous process with independent increments on some topological group G with a countable base. Stochastic continuity means that, for an arbitrary neighborhood of the identity U,

$$\lim_{h \to 0} P\{\omega : \xi_t^{-1}(\omega)\xi_{t+h}(\omega) \notin U\} = 0.$$

3. In the proof of Theorem 1 and Lemma 2, the following form of the generalized P. Lévy inequality (3) is used essentially.

Lemma 1. Let $x_1, x_2, \dots, x_n$ be independent random variables with values in a topological group G, and let A be some symmetric neighborhood of the identity of the group, i.e. $A = A^{-1}$. If, for any $j = 1, \dots, n$,

$$P\{x_j \cdots x_n \in A\} > 1 - \varepsilon$$

and

$$(1 - \varepsilon)/(1 + \varepsilon) > \varepsilon > 0,$$

then

$$P\{\text{one of } z_j \notin A^4,\ j = 1, \dots, n\} \leq$$

$$\leq (1 + \varepsilon)[1 - P\{z_n \in A\}]/P\{z_n \in A\}, \tag{1}$$

where $z_j = x_1 x_2 \cdots x_j$.

Proof. Denote by B the class of Borel sets on G generated by the closed sets. According to the generalized P. Lévy inequality (3), for all A, A', A_0 from B and $A' \times A^{-1} \subset A$, $A \ni e_1$ (the identity of G), and for any $\varepsilon > 0$, there exist $a_i \in G$, $a_n = e_1$, such that

$$P\{\text{one of } z_j a_j \notin A_0,\ j = 1, \dots, n\} \leq \frac{1 + \varepsilon}{Q_{\mu_n}(A)} P\{z_n \notin A'\}, \tag{2}$$

where $Q_{\mu_n}(A)$ is the concentration function of the distribution of the random variable z_n. The func-

tion $Q_\mu(A)$ of the measure μ is here defined by the equality

$$Q_\mu(A) = \sup_{x \in G} \mu(xA).$$

If in (2) the sets A' and A are set equal to A and $A_0 = A^2$, then, taking into account that

$$Q_{\mu_n}(A) = \sup_{x \in G} \mu_n(xA) \geq \mu_n(A) = P\{z_n \in A\},$$

we obtain

$$P\{\text{one of } z_j a_j \notin A^2\} \le (1+\varepsilon)[1 - P\{z_n \in A\}]/P\{z_n \in A\}.$$

Thus, to prove Lemma 1 it suffices to show that $a_j^{-1} \in A^2$. If we denote the distribution of the random variable $z_j a_j$ by μ_j, and the distribution of the random variable $x_{j+1} \dots x_n$ by ν_j, then

$$P\{z_n \in A\} = \mu_n(A) = \mu_j * a_j^{-1} * \nu_j(A) = \int_{y \in G} \nu_j(y^{-1} a_{jA})\, \mu_j(dy) \le Q_{\nu_j}(A). \quad (3)$$

Then the element a_j is determined from the condition

$$Q_{\nu_j}(A) \le (1+\varepsilon)\nu_j(a_{jA}).$$

But, by the condition of the lemma, $\nu_j(A) > 1 - \varepsilon$, whence $Q_{\nu_j}(A) > 1 - \varepsilon$. Therefore, if $a_{jA} \cap A = \emptyset$, then $\nu_j(a_{jA}) < \varepsilon$, and consequently it would have to be

$$\varepsilon < (1-\varepsilon)/(1+\varepsilon) < Q_{\nu_j}(A)/(1+\varepsilon) < \nu_j(a_{jA}) < \varepsilon.$$

Thus $a_{jA} \cap A = \emptyset$, and the element a_j^{-1}, by virtue of the symmetry of A, belongs to A^2.

4. A topological group with a countable base is metrizable. Denote by $\rho(x, y)$ the distance between elements x, y of G.

Theorem 1. *Every stochastically continuous, separable process $\xi_t(\omega)$ with independent increments, taking values in a topological group G with a countable base, almost everywhere has no discontinuities of the second kind.*

For the proof of the theorem it suffices to observe that the scheme of the proof of the analogous theorem in the classical case (4) is preserved also in our case, if instead of the norm of a number one takes the distance $\rho(e_1, x)$, $x \in G$, uses Lemma 1, and assumes the separability of $\xi_t(\omega)$ with respect to the class of closed sets in G. Applying the generalization of P. Lévy's theorem to metrizable groups (3), it is easy to obtain that for $\xi_t(\omega)$ there exists a stochastically equivalent separable process, continuous from the right. Therefore, in what follows we shall consider only processes continuous from the right.

Let $A_\varepsilon, B_\varepsilon, C_\varepsilon$ be sets from B for which $\rho(e_1, x) \ge \varepsilon$, $x \in A_\varepsilon, B_\varepsilon, C_\varepsilon$. The function $\nu(t, A_\varepsilon; \omega)$ is equal to the number of jumps $\xi_{\tau-0}^{-1}(\omega)\xi_\tau(\omega) \in A_\varepsilon$ for $a \le \tau \le t$. Denote by $[A_\varepsilon, B_\varepsilon, C_\varepsilon]$ some word of finite length written with the symbols $A_\varepsilon, B_\varepsilon, C_\varepsilon$. The set $M^{[u,v]}_{[A_\varepsilon, B_\varepsilon, C_\varepsilon]}$ denotes the set of ω for which the trajectories $\xi_t(\omega)$ have jumps in $A_\varepsilon, B_\varepsilon, C_\varepsilon$ during the time $[u, v]$ in the order of the arrangement of the symbols of the word $[A_\varepsilon, B_\varepsilon, C_\varepsilon]$.

As a consequence of Theorem 1 we obtain:

a) The quantities $\nu(t, A_\varepsilon; \omega)$ are almost everywhere finite and form an integer-valued process with independent increments, increasing by jumps of 1.

b) The sets $M^{[u,v]}_{[A_\varepsilon, B_\varepsilon, C_\varepsilon]}$ are measurable. The events $M^{[u,v]}_{[\ ]}$ and $M^{[u',v']}_{[\ ]'}$, for nonintersecting intervals $[u, v]$, $[u', v']$ and for any words $[\ ]$, $[\ ]'$, are independent.

5. **Lemma 2.** *The distribution of the random variable $\nu(t, A_\varepsilon; \omega)$ is continuous in t.*

Proof. Since $\nu(t, A_\varepsilon; \omega)$ have independent increments, to prove the lemma it is necessary to show that the probability of the event

$$\{\omega : \ \nu(t+h, A_\varepsilon; \omega) - \nu(t, A_\varepsilon; \omega) \geq 1\} = M_h(A_\varepsilon) \to 0$$

as $h \to 0$. By the metrizability of G there is a symmetric closed neighborhood of the identity A such that $\rho(e_1, x^8) < \varepsilon'' < \varepsilon'$ for $x \in A$. We shall show that the probability of the event

$$\{\omega : \xi_t^{-1}(\omega)\xi_\tau(\omega) \in A^4\}$$

for $t \leq \tau \leq t+h$ tends to 1 as $h \to 0$. Divide the interval $[t, t+h]$ by the points $t_i^{(n)}$ into n parts. Then almost everywhere

$$\{\omega : \ \xi_t^{-1}(\omega)\xi_\tau(\omega) \in A^4\} = \Omega \setminus \lim_{\Delta \to 0} \{\omega : \ \text{one of } \xi_t^{-1}(\omega)\xi_{t_i^{(n)}}(\omega) \notin A^4\}, \qquad (4)$$

where $\Delta = \max(t_{i+1}^{(n)} - t_i^{(n)})$. By virtue of the stochastic continuity of $\xi_t(\omega)$, for any $\varepsilon > 0$ one can choose $h_\varepsilon > 0$ so that for all t in $[a, b]$ $P\{\xi_t^{-1}(\omega)\xi_{t+h}(\omega) \in A\} > 1 - \varepsilon$, if $h < h_\varepsilon$. Applying Lemma 1 to the random variables $\xi^{-1}_{t_i^{(n)}}(\omega)\xi_{t_{i+1}^{(n)}}(\omega)$ for $h < h_\varepsilon$, we then have

$$P\{\text{one of } \xi_t^{-1}(\omega)\xi_{t_i^{(n)}}(\omega) \notin A^4\} \leq (1+\varepsilon)\,\frac{1 - P\{\xi_t^{-1}(\omega)\xi_{t+h'}(\omega) \in A\}}{P\{\xi_t^{-1}(\omega)\xi_{t+h}(\omega) \in A\}}.$$

Letting $\Delta \to 0$ and taking (4) into account, we obtain

$$\lim_{h \to 0} P\{\omega : \ \xi_t^{-1}(\omega)\xi_\tau(\omega) \in A^4\} = 1, \qquad t \leq \tau \leq t+h. \qquad (5)$$

The following identity holds:

$$M_h(A_{\varepsilon'}) = M_h(A_{\varepsilon'}) \cap \{\omega : \ \xi_t^{-1}(\omega)\xi_\tau(\omega) \in A^4\} + M_h(A_{\varepsilon'}) \cap \{\omega : \ \xi_t^{-1}(\omega)\xi_\tau(\omega) \in \overline{A^4}\}, \qquad t \leq \tau \leq t+h.$$

By (5), as $h \to 0$,

$$P\{M_h(A_{\varepsilon'}) \cap \{\omega : \xi_t^{-1}(\omega)\xi_\tau(\omega) \in \overline{A^4}\}\} \to 0.$$

On the other hand, it is evident that if A is symmetric and $A^8 \cap A_{\varepsilon'} = \emptyset$, then

$$M_h(A_{\varepsilon'}) \cap \{\omega : \ \xi_t^{-1}(\omega)\xi_\tau(\omega) \in A^4\} = \emptyset.$$

Lemma 3. Let $A_{\varepsilon_1}, B_{\varepsilon_2}, \dots, D_{\varepsilon_k}$, $\varepsilon_i > 0$, be k arbitrary sets from $\mathcal{B}$. Then the vector

$$\{v(t, A_{\varepsilon_1}; \omega),\ v(t, B_{\varepsilon_2}; \omega), \dots,\ v(t, D_{\varepsilon_k}; \omega)\}$$

is a stochastically continuous process with independent increments on the k-dimensional lattice.

Proof. For simplicity we consider the case of two sets A_{ε_1} and B_{ε_2}. We show the independence of

$$v(t_1 + \Delta_1, \omega) - v(t_1, \omega) = \Delta_1 v$$

and

$$v(t_2 + \Delta_2, \omega) - v(t_2, \omega) = \Delta_2 v$$

on nonintersecting intervals. Since the components of $\Delta_1 v$ take integer values, $\Delta_1 v$ takes its values from the lattice (m_1, n_1). The set of ω for which $\Delta_1 v = (m_1, n_1)$ is equal to

$$\bigcup_{[\,]} M^{[t_1, t_1+\Delta_1]}_{[A_{\varepsilon_1}, B_{\varepsilon_2}, C_\varepsilon]}.$$

Here the union is taken over all words that can be formed from the symbols $A_{\varepsilon_1}, B_{\varepsilon_2}, C_\varepsilon = A_{\varepsilon_1} \cap B_{\varepsilon_2}$, and in which the sum of the numbers of the letters A_{ε_1} and C_ε is equal to m_1, while the sum of the numbers of the letters B_{ε_2} and C_ε is equal to n_1. Similarly, the set of ω for which $\Delta_2 v = (m_2, n_2)$ is

$$\bigcup_{[\,]} M^{[t_2, t_2+\Delta_2]}_{[A_{\varepsilon_1}, B_{\varepsilon_2}, C_\varepsilon]}.$$

Since

$$M^{[u,v]}_{[\,]} \cap M^{[u',v']}_{[\,]'} = \emptyset$$

for different $[\,]$, $[\,]'$, and b) holds, the quantities $\Delta_1 v$ and $\Delta_2 v$ are independent.

To prove the stochastic continuity of $v(t, \omega)$, by the independence of increments of $v(t, \omega)$ it is enough to show that

$$P\{\Delta v = v(t + \Delta, \omega) - v(t, \omega) \neq (0, 0)\} \to 0 \quad \text{as } \Delta \to 0.$$

But

$$0 \leq P\{\Delta v \neq (0, 0)\} \leq P\{M_\Delta(A_\varepsilon)\} + P\{M_\Delta(B_\varepsilon)\}.$$

By Lemma 2, the right-hand side of the inequality tends to zero as $\Delta \to 0$.

6. To prove Theorem 4 we shall need distributions that are a generalization of the Poisson distributions to the k-dimensional lattice.

Definition. A distribution P on the k-dimensional lattice is called multidimensional Poisson if its generating function has the form

$$e^{-D} \exp \left\{ \sum_{i=1}^{k} a_i x_i + \sum_{i_1 \neq i_2} a_{i_1,i_2} x_{i_1} x_{i_2} + \cdots + \sum_{i_1 \neq i_2 \neq \cdots \neq i_k} a_{i_1,\ldots,i_k} x_{i_1} \cdots x_{i_k} \right\},$$

where $a_i \geq 0,\ a_{i_1,i_2} \geq 0, \ldots,\ a_{i_1,\ldots,i_k} \geq 0$, and

$$D = \sum a_i + \sum a_{i_1,i_2} + \cdots + \sum a_{i_1,\ldots,i_k}.$$

If all $a_{i_1,i_2}, \ldots, a_{i_1,i_2,\ldots,i_k}$ are equal to zero, then the components of P are independent and are Poisson distributed with parameters a_i.

Lemma 4. If the vector $(\xi_1, \xi_2, \ldots, \xi_k)$ has a multidimensional Poisson distribution and the sum of the coordinates $\xi_1 + \xi_2 + \cdots + \xi_k = \eta$ is Poisson distributed, then the quantities $\xi_1, \xi_2, \ldots, \xi_k$ are mutually independent.

Proof. Since η is Poisson distributed, we have

$$P\{\eta = 0\}/P\{\eta = 1\} = -\ln P\{\eta = 0\}. \tag{6}$$

But

$$P\{\eta = 0\} = e^{-D}, \qquad P\{\eta = 1\} = e^{-D}\left(\sum a_i\right), \qquad \sum a_i + \sum a_{i_1,i_2} + \cdots + \sum a_{i_1,\ldots,i_k} = D = -\ln P\{\eta = 0\}.$$

Substituting these expressions into (6), we obtain

$$\sum a_i + \sum a_{i_1,i_2} + \cdots + \sum a_{i_1,\ldots,i_k} = \sum a_i,$$

i.e.

$$\sum a_{i_1,i_2} + \cdots + \sum a_{i_1,\ldots,i_k} = 0.$$

Since $a_{i_1,i_2} \geq 0, \ldots, a_{i_1,\ldots,i_k} \geq 0$, all these numbers are equal to zero. Consequently, the ξ_i are mutually independent and are Poisson distributed.

Let $\eta_t(\omega)$ be a stochastically continuous process with independent increments on the k-dimensional lattice. If for each t the distribution of $\eta_t(\omega)$ is multidimensional Poisson, then $\eta_t(\omega)$ will be called a multidimensional Poisson process.

The following propositions generalize the well-known properties of the trajectories of a Poisson process.

Theorem 2. If the process $\eta_t(\omega)$ is separable, then almost all its trajectories have a finite number of jumps. Each of them is written in the form $[\alpha_1, \ldots, \alpha_k]$, where $\alpha_i,\ i = 1, \ldots, k$, may take the value either 0 or 1.

The converse assertion:

Theorem 3. If almost all trajectories of a stochastically continuous process $\eta_t(\omega)$ with independent increments have a finite number of jumps, each of which is written in the form $[\alpha_1, \ldots, \alpha_k]$, then $\eta_t(\omega)$ is a separable multidimensional Poisson process.

The following proposition is a generalization of K. Ito' s theorem.

Theorem 4. Let $\xi_t(\omega)$ be a stochastically continuous, separable process with independent increments, taking values in an arbitrary topological group G with a countable base. Then for any sets $A_{\varepsilon_1}, B_{\varepsilon_2}, \ldots, D_{\varepsilon_k}$, $\varepsilon_i > 0$, from B, the vector

$$\{\nu(t, A_{\varepsilon_1}; \omega), \nu(t, B_{\varepsilon_2}; \omega), \ldots, \nu(t, D_{\varepsilon_k}; \omega)\}$$

will be a random variable with a multidimensional Poisson distribution. If the sets $A_{\varepsilon_1}, B_{\varepsilon_2}, \ldots, D_{\varepsilon_k}$ are mutually disjoint, then the random variables

$$\nu(t, A_{\varepsilon_1}; \omega), \nu(t, B_{\varepsilon_2}; \omega), \ldots, \nu(t, D_{\varepsilon_k}; \omega)$$

are mutually independent.

Proof. The first part of the assertion of the theorem is a consequence of Theorems 1, 3 and Lemma 3.

If $A_{\varepsilon_1}, B_{\varepsilon_2}, \ldots, D_{\varepsilon_k}$ do not intersect, then

$$A_{\varepsilon_1} \cup B_{\varepsilon_2} \cup \cdots \cup D_{\varepsilon_k} = U_\varepsilon,$$

where

$$\varepsilon = \min_i \varepsilon_i$$

and the sum

$$\nu(t, A_{\varepsilon_1}; \omega) + \cdots + \nu(t, D_{\varepsilon_k}; \omega)$$

is equal to

$$\nu(t, U_\varepsilon; \omega).$$

By Lemmas 2, 3 and Theorem 3, the quantity $\nu(t, U_\varepsilon; \omega)$ is Poisson distributed. Thus, from Lemma 4 we obtain that the quantities

$$\nu(t, A_{\varepsilon_1}; \omega), \ldots, \nu(t, D_{\varepsilon_k}; \omega)$$

are mutually independent.

In conclusion we note that the metrizability of G was used essentially only in the proof of Theorem 1, and also for the transition to a stochastically equivalent process $\xi_t(\omega)$ continuous from the right.

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Note: Figure translations are in progress. See original paper for figures.