ON THE COMPLETENESS OF A SYSTEM OF EIGENVECTORS AND ASSOCIATED VECTORS OF AN OPERATOR PENCIL

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Abstract

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ON THE COMPLETENESS OF A SYSTEM OF EIGENVECTORS AND ASSOCIATED VECTORS OF AN OPERATOR PENCIL

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The problem of the completeness of the system of eigenvectors and associated vectors of a polynomial operator pencil was first considered by M. V. Keldysh ([1]). In a number of works ([2–5]) the authors succeeded in weakening the restrictions on the leading part of the pencil imposed by the theorem of M. V. Keldysh, and in extending this theorem to pencils of a more general structure. In the present note a number of criteria are formulated for the completeness of the system of eigenvectors and associated vectors of certain classes of operator pencils, which are established by means of the method of estimating resolvents of linear operators proposed by V. I. Macaev ([6]). In the case of a linear pencil they were obtained by V. I. Macaev ([7]).

In what follows, according to ([8]), $\mathfrak{S}_\infty$ denotes the Banach space of all linear completely continuous operators acting in a separable Hilbert space $\mathfrak{H}$ with the usual norm

$$\|A\|_\infty = \sup_{f\in\mathfrak{H}}(\|Af\|/\|f\|)$$

($A \in \mathfrak{S}_\infty$); with each such operator A there is associated the sequence of its s-numbers, defined as the eigenvalues of the operator $(A^*A)^{1/2}$, numbered in decreasing order with multiplicities taken into account (if $r = \dim(A^*A)^{1/2} < \infty$, then $s_j(A) = 0$ ($j = r+1, ...$)); and the function $\nu(t, A)$ ($t > 0$), giving the number of numbers $s_n(A)$ greater than t^{-1}. $\mathfrak{S}_p$ ($p > 0$) denotes the class of operators A ($A \subset \mathfrak{S}_\infty$) for which

$$\sum_{k=1}^{\infty} s_k^p(A) < \infty,$$

and $\mathfrak{S}_\omega$ is the class of operators A ($A \subset \mathfrak{S}_\infty$) for which

$$\sum_{k=1}^{\infty}(2k-1)^{-1}s_k(A) < \infty.$$

Lemma 1. Let in the operator pencil

$$L(\lambda) = 1 - \sum_{i=1}^{n} \lambda^i B_i$$

the operators $B_i \in \mathfrak{S}_\infty$, and let the elements $f, g_i \in \mathfrak{H}$ $(i = 0, 1, \dots, n-1)$ be such that the function

$$F(\lambda) = \left(L^{-1}(\lambda) f,\ \sum_{i=0}^{n-1} \overline{\lambda}^i g_i \right)$$

is entire. Then for sufficiently large $r = |\lambda|$ the estimate

$$\ln |F(\lambda)| \leqslant (n-1) \ln r + \alpha \sum_{i=1}^{n} \int_0^{(n+2)(2r)^i} t^{-1} \nu(t, B_i)\, dt.$$

With the aid of a lemma of M. V. Keldysh ([8]) and Lemma 1, an estimate in terms of the s-numbers of the operator H is found for the function $\ln |F(\lambda)|$ corresponding to the operator pencil

$$L(\lambda) = 1 - \sum_{i=0}^{n} \lambda^i A_i H^i, \tag{1}$$

in which the operators $A_i \in \mathfrak{S}_\infty$ $(i = 0, 1, \dots, n-1)$, $A_n = 1$, and H is a complete normal operator with spectrum on the rays $l_q = \{\lambda : \arg \lambda = q\pi/n\}$ $(q = 0, \dots, 2n-1)$,

$$\ln |F(\lambda)| \leq a_1 \ln r \cdot \nu(a_2 r, H). \tag{2}$$

With the aid of the Poisson integral one obtains an estimate for the function $\ln |F(\lambda)|$, corresponding to the pencil (1), in terms of the s-numbers of the operators A_i, which is established by

Lemma 2. If $f, g_i \in \mathfrak{H}$ $(i = 0, 1, \dots, n-1)$ are such that $F(\lambda)$ is a holomorphic function for $\lambda \in l_q$, then the inequality

$$\ln |F(\lambda)| \leq \frac{C_1}{|\sin n\varphi|} \left(\ln \frac{1}{|\sin n\varphi|} + \sum_{i=0}^{n-1} \int_0^{C_2 |\sin n\varphi|^{-1}} t^{-1} \nu(t, A_i)\, dt + \ln r \right)$$

$$(\varphi = \arg \lambda,\ r > r_0). \tag{3}$$

From Lemma 2 and Levinson' s lemma ([9]) it follows that

Theorem 1. If in the pencil (1) the operators $A_i \subset \mathfrak{S}_\infty$ $(i = 0, 1, \dots, n-1)$, then the system of eigenvectors and associated vectors of the pencil is n-fold complete.

Analogously to Theorem 1 one establishes the assertion: if in the pencil $L(\lambda) = 1 - A_0 - \lambda H - \frac{1}{\lambda} G$ the operators $H = H^*$, $G = G^* \in \mathfrak{S}_\infty$ are complete, and the operator $A_0 \in \mathfrak{S}_\omega$, then the system of eigenvectors and associated vectors of the pencil is two-fold complete.

Using estimates (2), (3) and Ahlfors' inequality on distortion under a conformal mapping of a curvilinear strip onto a strip ([10]), the following is proved.

Theorem 2. The system of eigenvectors and associated vectors of the operator pencil (1) is n-fold complete if the operator $1 - A_0$ is invertible and

$$\lim_{m\to\infty} \left| \sum_{i=0}^{n-1} \left\{ \sum_{j=1}^{m} (2j-1)^{-1} \cdot s_j(A_i) \right\} \right| \Big/ \ln s_m^{-1}(H) = 0.$$

This theorem contains as a special case Theorem 1 under the condition that the operator $1 - A_0$ is invertible, and the theorem on the completeness of the eigenvectors and associated vectors of the pencil (1) under the condition $\lim_{r\to\infty} \ln \nu(r, H)/\ln r < \infty$, which is a generalization of M. V. Keldysh' s theorem.

For the pencil

$$L(\lambda) = 1 - \sum_{i=0}^{n} \lambda^i A_i H^i - \sum_{l=1}^{N} \sum_{k=1}^{n_l} \frac{A_{l,k} H_l^k}{(\lambda - \lambda_l)^k}, \tag{4}$$

in which the operators A_i, H are subject to the same restrictions as in (1), the operators $A_{l,k}$ $(l = 1, \dots, N;\ k = 1, \dots, n_l - 1) \in \mathfrak{S}_\infty$, $A_{l,n_l} = e^{i\theta_l} 1$,

and the operators H_l are complete normal operators with spectrum on the rays

$$l_{ql}\{\lambda : \arg\lambda = q_l \pi / n_l\} \quad (q_l = 0, 1, \dots, 2n_l - 1).$$

With the aid of an estimate of the ratio of Poisson kernels for sufficiently distant points of an annulus having the same modulus, one establishes

Theorem 3. The system of eigenvectors and associated functions of the pencil

$$(4) \qquad n = \sum_{l=1}^{N} n_l$$

is multiply complete if

$$\lim_{r\to\infty} \ln\nu(r,H)/\ln r < \infty, \qquad \lim_{r\to\infty} \ln\nu(r,H_l)/\ln r < \infty \quad (l=1,\dots,N).$$

Lemma 3. Let, in the pencil (1), the operator $1-A_0$ be invertible, and let the elements $f, g_i \in \mathfrak{H}$ be such that $F(\lambda)$ is an entire function; then for sufficiently small

for $|\varphi|$ the estimate holds

$$\ln\left|\frac{F(\lambda)}{(\lambda+1)^{n+1}}\right| \ll \ln\frac{d_1}{|\sin n\varphi|}+\left(1+\frac{d_2}{|\sin n\varphi|^{n+1}}\right)\sum_{i=0}^{n-1}\int_0^{d_3 r/|\sin n\varphi|} t^{-1}\nu(t,A_{iH})\,dt.$$

Analogous estimates also hold for the remaining rays $\arg\lambda = \pi q/n$ $(q=1,\dots,2n-1)$.

With the aid of Lemma 3, Varshavskii's theorem on the asymptotics of a function conformally mapping a curvilinear strip onto a strip (11), and estimates (2) and (3), one proves

Theorem 4. *If the operator $1-A_0$ is invertible, then the system of eigenvectors and associated vectors of the operator pencil (1) is n-fold complete, provided the operators $A_{iH} \in \mathfrak{S}_\omega$ $(i=0,\dots,n-1)$ and*

$$\lim_{m\to\infty}\sum_{i=0}^{n-1}\left\{\sum_{j=1}^{m}(2j-1)^{-1}s_j(H)\right\}\Bigg/\left\{\ln\left(\sum_{i=0}^{n-1}\left\{\sum_{j=m}^{\infty}(2j-1)^{-1}s_j(A_{iH})\right\}\right)\right\}^{-1}=0.$$

In particular, this implies a completeness theorem, if the operators $A_{iH}\in\mathfrak{S}_p$, close to the theorem from (4).

Let in the pencil

$$L(\lambda)=1-\sum_{i=0}^{n}\lambda^i A_i H^{i+\alpha} \tag{5}$$

the operators A_i, H have the previous properties, $\alpha>0$. For such a pencil, analogously to Theorems 2 and 4, Theorems 5 and 6 are established.

Theorem 5. *The system of eigenvectors and associated vectors of the pencil (5) is n-fold complete, if the operator $1-A_0H^\alpha$ is invertible and*

$$\lim_{m\to\infty}\sum_{i=0}^{n-1}\left(\sum_{j=1}^{m}(2j-1)^{-1}s_j(A_i)\right)/s_m^{-\alpha}(H)=0.$$

Theorem 6. *The system of eigenvectors and associated vectors of the pencil (5) is n-fold complete, if the operators* $A_i H^{1+\alpha} \in \mathfrak{S}_\omega$ $(i = 0, \dots, n-1)$, *the operator* $1 - A_0 H^\alpha$ *is invertible and*

$$\lim_{m\to\infty} \sum_{i=0}^{n-1} \left\{ \sum_{j=1}^{m} (2j-1)^{-1} s_j(A_i) \right\} \Big/ \left\{ \sum_{i=0}^{n-1} \left(\sum_{j=m}^{\infty} (2j-1)^{-1} s_j(A_i H^{1+\alpha}) \right) \right\}^{-\alpha} = 0.$$

The theorems presented are also valid in the case when, instead of the pencils (1), (4), (5), one considers the pencils

$$L(\lambda) = 1 - \sum_{i=0}^{n} \lambda^i H^i A_i; \tag{1'}$$

$$L(\lambda) = 1 - \sum_{i=0}^{n} \lambda^i H^i A_i - \sum_{l=1}^{N} \sum_{k=1}^{n_l} \frac{H_l^k A_{l,k}}{(\lambda - \lambda_l)^k}; \tag{4'}$$

$$L(\lambda) = 1 - \sum_{i=0}^{n} \lambda^i H^{i+\alpha} A_i. \tag{5'}$$

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Note: Figure translations are in progress. See original paper for figures.