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Abstract

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MATHEMATICS

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APPROXIMATE SOLUTION OF SINGULAR INTEGRAL EQUATIONS BY THE METHOD OF MECHANICAL QUADRATURES

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A number of works have been devoted to the approximate solution of singular integral equations without regularizing them,* in which, first of all, well-known direct methods (mainly of projection type) were transferred from regular integral equations to singular ones and justified. However, the study of the method of mechanical quadratures has essentially not been carried out up to now.** Below, for singular integral equations with a Cauchy-type kernel, various computational schemes of the method of mechanical quadratures are indicated, and an effective way of justifying this method in the case of equations of normal type is given.

§ 1. **Computational schemes.** Consider integral equations of the form

$$Ax \equiv a(t)x(t) + \frac{b(t)}{\pi i}\int_\gamma \frac{x(\tau)\,d\tau}{\tau - t} + \frac{\lambda}{2\pi i}\int_\gamma h(t,\tau)x(\tau)\,d\tau = f(t), \qquad (1,1)$$

where the coefficients and the right-hand side are continuous functions on the circle γ of unit radius with center at the origin, and λ is a complex parameter.

a) If the index of equation (1,1) is equal to zero, then its approximate solution is sought in the form of the interpolation polynomial

$$\tilde{x}(t) = \frac{1}{2n+1}\sum_{k=0}^{2n} c_k \sin(2n+1)\frac{s - s_k}{2} \operatorname{cosec}\frac{s - s_k}{2}. \qquad (1,2)$$

The coefficients $\{c_j\}$ are determined from the system of linear algebraic equations

$$a_{jc}j + \frac{b_j}{2n+1}\sum_{k=0}^{2n}\alpha_{jk}c_k + \frac{\lambda}{2n+1}\sum_{k=0}^{2n}h_{jk}c_k = f_j \quad (j = 0, \dots, 2n), \qquad (1,3)$$

where $a_j = a(t_j)$; $b_j = b(t_j)$; $f_j = f(t_j)$; $h_{jk} = h(t_j, t_k)t_k$; $\alpha_{jk} = 1 - i\beta_{jk}$; $t = e^{is}$; $t_j = e^{is_j}$; $s_j = 2j\pi/(2n+1)$; $\beta_{jk} = \mathrm{tg}(s_j - s_k)/4$, $j-k$ even; $\beta_{jk} = \mathrm{ctg}(s_k - s_j)/4$, $j-k$ odd.

b) For an arbitrary index $\chi = \mathrm{ind}\, A$, for definiteness one seeks such solutions of equation (1,1) for which the Cauchy-type integral

$$\frac{1}{2\pi i} \int_\gamma \frac{x(\tau)\, d\tau}{\tau - z}$$

has at infinity a zero of the highest possible order (1). Then the approximate solution is found in the form of a polyterm

$$\tilde{x}(t) = \sum_{k=0}^{n} a_k t^k + \sum_{k=-n}^{-1} a_k t^{k-\chi}, \tag{1,4}$$

whose coefficients are found from the system of equations*

$$(a_j+b_j)\sum_{k=0}^{n} a_k t_j^k + (a_j-b_j)\sum_{k=-n}^{-1} a_k t_j^{k-\chi} + \frac{\lambda}{2n+1}\sum_{k=-n}^{n} c_{jk} a_k = f_j \quad (j = 0, \dots, 2n) \tag{1,5}$$

where

$$c_{jk} = \sum_{r=0}^{2n} h_{jr} t_r^{k-\sigma},$$

with $\sigma = 0$ for $k \geq 0$ and $\sigma = \chi$ for $k < 0$.

§ 2. Justification of the method of mechanical quadratures.

Let the functions a, b, h (in both arguments), and f belong to the class $H_\alpha^{(r)}$ – that is, have r $(r = 0, 1, \dots)$ continuous derivatives satisfying a Hölder condition with exponent α $(0 < \alpha \leq 1)$, and suppose that $a^2 - b^2 \neq 0$ nowhere on γ. Under these conditions equation (1,1) can be regarded as a normally solvable

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* In work (1) the main results of most of the works that appeared up to and including 1963 are presented.

** A number of particular results on the method of mechanical quadratures are available in works (2,3) for integral equations with a Cauchy kernel of the first kind, posed along a segment of the real axis, and also in our work (4) for equations with a Hilbert kernel.

linear operator equation in the Banach space of functions $X = H_\beta$ $(0 < \beta < \alpha)$, with norm (5)

$$\|x\| = M(x) + H(x;\beta) \equiv \max_t |x(t)| + \sup_{t' \neq t''} \frac{|x(t') - x(t'')|}{|t' - t''|^\beta} \quad (t',\, t'' \in \gamma).$$

Theorem 1. *If $\chi = 0$ and λ is not a characteristic value (6) of the operator A, then for n such that*

$$q = (A_1 \ln n + B_1) n^{-r-\alpha+\beta} < 1, \tag{2,1}$$

*the system of equations (1,3) has a unique solution for any right-hand side. The approximate solutions (1,2) converge to the exact solution $x(t)$ of equation (1,1) with rate**:*

$$\|x - \tilde{x}\| \leq (A_2 \ln n + B_2) n^{-r-\alpha+\beta} \quad (r \geq 0,\; 0 < \beta < \alpha \leq 1). \tag{2,2}$$

Proof. To each function $x(t) \in H_\beta$ we associate the functions $x^+(z)$ and $x^-(z)$, analytic respectively inside and outside γ, and related to $x(t)$ by the Plemelj-Sokhotski formulas (5):

$$x^+ - x^- = x, \qquad x^+ + x^- = Sx = \frac{1}{\pi i} \int_\gamma \frac{x(\tau)\, d\tau}{\tau - t}. \tag{2,3}$$

Since (5) the function $(a-b)/(a+b)$ is representable in the form

$$\frac{a-b}{a+b} = \frac{\psi^+}{\psi^-}, \qquad \psi^\pm(t) = \exp \theta^\pm(t), \qquad \theta(z) = \frac{1}{2\pi i} \int_\gamma \frac{\ln(a-b)/(a+b)}{\tau - z}\, d\tau, \tag{2,4}$$

equations (1,1) and (1,3) reduce respectively to the following equivalent equations:

$$\psi^- x^+ - \psi^+ x^- + \lambda d T x = y, \tag{2,5}$$

$$\psi_j^- \sum_{k=0}^{n} a_k t_j^k + \psi_j^+ \sum_{k=-n}^{-1} a_k t_j^k + \frac{\lambda d_j}{2n+1} \sum_{k=0}^{2n} h_{jk} c_k = y_j \quad (j = 0, \dots, 2n), \tag{2,6}$$

where $y = df,\ y_j = y(t_j),\ \psi_j^\pm = \psi^\pm(t_j),\ c_j = \sum_{k=-n}^{n} a_k t_j^k = \tilde{x}(t_j),\ d = \dfrac{\psi^-}{a+b},$

$$a_k = \frac{1}{2n+1} \sum_{j=0}^{2n} c_j t_j^{-k}, \qquad Tx \equiv Thx = \frac{1}{2\pi i} \int_\gamma h(t,\tau) x(\tau)\, d\tau.$$

Using the properties $(1,5)$ of the operators S and T and taking into account the relations $(2,3)$ and $(2,4)$, equation $(2,5)$ can be regarded in the space $X = \{x^+ - x^-\} = H_\beta$ as a functional equation of the form

$$Kx = Gx + \lambda Ux = y \qquad (x \in X,\ y \in X), \tag{2,7}$$

where $Gx = \psi^- x^+ - \psi^+ x^-$, $Ux = dTx \equiv Uhx$, with G linear and U a completely continuous operator.

Let $P = P_n$ be the operator of projection onto the set of interpolation polynomials of degree n at the nodes $t_k = t_k^{(n)}$ $(k = 0, \dots, 2n)$. Denote by P_τ the operator P applied with respect to the variable τ. Then system $(2,6)$ can be written as the linear equation

$$\widetilde{K}\tilde{x} = \widetilde{G}\tilde{x} + \lambda \widetilde{U}\tilde{x} = \tilde{y} \qquad (\tilde{x} \in \widetilde{X},\ \tilde{y} \in \widetilde{X}), \tag{2,8}$$

$$\widetilde{G}\tilde{x} = PG\tilde{x}, \qquad \widetilde{U}\tilde{x} = P_t U P_\tau (h\tilde{x}), \qquad \tilde{y} = Py$$

in the $(2n+1)$-dimensional space of polynomials

$$\widetilde{X} = \{\tilde{x}^+ - \tilde{x}^-\} = \left\{ \sum_{k=-n}^{n} \alpha_k t^k \right\}$$

with the same norm as above.

Following (17), consider in the space X the boundary-value problem

$$K_{nx} = G_{nx} + \lambda Ux = y, \qquad G_{nx} = \psi_n^- x^+ - \psi_n^+ x^-, \tag{2,9}$$

where $\psi_n = \psi_n^+ - \psi_n^-$ is the polynomial of best uniform approximation of degree n to the function $\psi(t) = \exp \theta(t)$. Therefore, for $q_1 = A_3 \|K^{-1}\| n^{-r-\alpha+\beta} < 1$, the operator K_n has an inverse, and

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* For $\chi = 0$ the systems (1,3) and (1,5) are equivalent, but from the practical point of view the first system is

** By A_k, B_k we denote well-defined constants independent of n.

$$\|K_n^{-1}\| \leq \|K^{-1}\|(1-q_1)^{-1} \leq B_3,$$
$$\|x - x^{(n)}\| \equiv \|K^{-1}y - K_n^{-1}y\| \leq \|q_1\|y\|\|K_n^{-1}\|. \qquad (2.10)$$

Now for equation (2.9) consider the system of the method of mechanical quadratures in operator form:

$$\widetilde{K}_n\tilde{x} = \widetilde{G}_n\tilde{x} + \lambda\widetilde{U}\tilde{x} = \tilde{y} \qquad (\tilde{x} \in \widetilde{X},\ \tilde{y} \in \widetilde{X}). \qquad (2{,}11)$$

Putting $R = E - P$, for any $\tilde{x} \in \widetilde{X}$ we find

$$\|\widetilde{K}_n\tilde{x} - P_{nK}n\tilde{x}\| = |\lambda|\,\|P_t[UR_\tau(h\tilde{x})]\| \leq \frac{A_4 \ln n + B_4}{n^{r+\alpha-\beta}}\|\tilde{x}\| \equiv \varepsilon_1\|\tilde{x}\| \quad (n = 1, 2, \ldots). \qquad (2{,}12)$$

Let $\tilde{c} = G_n\tilde{x} + \lambda\widetilde{\varphi}$, where $\widetilde{\varphi}$ is the polynomial of best uniform approximation for the element $\varphi = U\tilde{x}$. Since $\varphi \in \widetilde{H}_\alpha^{(r)}$ and $\widetilde{H}(\varphi^{(r)}; \alpha) \leq A_5\|\tilde{x}\|$, for any $\tilde{x} \in \widetilde{X}$ we have

$$\|K_n\tilde{x} - \tilde{c}\| = |\lambda|\|\varphi - \widetilde{\varphi}\| \leq B_5 n^{-r-\alpha+\beta}\|\tilde{x}\| \equiv \varepsilon_2\|\tilde{x}\| \quad (n = 1, 2, \ldots). \qquad (2{,}13)$$

Lemma. If n is such that

$$q_2 = \max\{q_1, (\varepsilon_1 + \varepsilon_2\|R\|)\|K_n^{-1}\|\} \leq (A_6 \ln n + B_6)n^{-r-\alpha+\beta} < 1, \qquad (2{,}14)$$

then the operator $\widetilde{K}_n$ has a bounded inverse, and for the solutions of equations $(2, 9)$ and $(2, 11)$ the estimate holds:

$$\|x^{(n)} - \tilde{x}^{(n)}\| \equiv \|K_n^{-1}y - \widetilde{K}_n^{-1}\tilde{y}\| \leq (q_2\|y\| + \|Ry\|)\frac{\|K_n^{-1}\|}{1 - q_2} \leq \frac{A_7 \ln n + B_7}{n^{r+\alpha-\beta}}. \qquad (2{,}15)$$

Further, since $\|\widetilde{K} - \widetilde{K}_n\| \leq A_3\|P\|n^{-r-\alpha+\beta} \equiv q_3\|\widetilde{K}_n^{-1}\|^{-1}$, it follows that for $q_4 = \max_{1\leq i\leq 3}\{q_i\} \leq q < 1$, equation $(2, 8)$, and hence also system $(1, 3)$, has a unique solution, and moreover

$$\|\widetilde{K}^{-1}\| \leq \|\widetilde{K}_n^{-1}\|(1-q)^{-1},$$

$$\|\tilde{x} - \tilde{x}^{(n)}\| \equiv \|\widetilde{K}^{-1}\tilde{y} - \widetilde{K}_n^{-1}\tilde{y}\| \leq q(\|y\| + \|Ry\|)\|\widetilde{K}^{-1}\|. \qquad (2.16)$$

Estimate (2.2) follows from inequalities (2.10), (2.15), and (2.16).

For an arbitrary index $\varkappa = \operatorname{ind} A$, the following is true.

Theorem 2. Let λ be a noncharacteristic (6) value of the operator A. If equation (1.1) and the boundary-value problem

$$(a+b)x_1^+ - (a-b)x_1^- t^{-\varkappa} + \lambda Th(x_1^+ - \tau^{-\varkappa} x_1^-) = f$$

$$(x_1^+ = x^+,\ x_1^- = x^{-\varkappa}) \tag{2.17}$$

are equivalent*, then for

$$q = q(\varkappa) = (A_8 \ln n + B_8) n^{-r-\alpha+\beta} < 1 \tag{2.18}$$

the system (1.5) is uniquely solvable and the approximate solution (1.4) converges to the exact solution at the rate

$$\|x - \tilde{x}\| \leq (A_9 \ln n + B_9) n^{-r-\alpha+\beta}. \tag{2.19}$$

The proof reduces to the proof of Theorem 1.

In connection with Theorems 1 and 2, let us note that all the constants used above, except for the norm $\|K^{-1}\|$, are either determined exactly or estimated from above**. On the other hand, if as the basic space one takes the space (1) $X = W$, then in the estimates given above one may put $\beta = 0$. However, in this case, unlike $X = H_\beta$, it is not possible to find a simple expression for the indicated constants.

Denote by $\Delta_n(\lambda)$ the determinants of the systems (1.3) and (1.5). The roots of the equation $\Delta_n(\lambda) = 0$ will be called the approximate characteristic values of the n-th approximation.

Theorem 3. The characteristic values of the operator A, and only they, can be obtained as limits of all possible sequences of approximate characteristic values.

The proof is carried out with the aid of works (1,6).

In conclusion, let us note that the method presented here for investigating the method of mechanical quadratures can also be applied to other classes of singular integral equations. This can be done either directly, or by reducing the equation under consideration to the form studied here***. In particular, with a corresponding choice of the initial spaces, results analogous to those given above are also valid for the computational schemes of works (2–4); moreover, in the case of equations of the first kind (2,3), the proofs are considerably simplified and can be carried out directly by means of the lemma with $A = K = K_n$.

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REFERENCES

[1] V. V. Ivanov, *Itogi nauki*, ser. Mathematical Analysis, 1963, Moscow, 1965, p. 125.
[2] A. I. Kalandia, DAN, 125, No. 4, 715 (1959).
[3] H. Brakhage, ZAMM, 41, Sonderheft, 12 (1961).
[4] B. G. Gabdulkhayev, *Izv. vyssh. uchebn. zaved.*, Mathematics, No. 5 (48), 43 (1965).
[5] N. I. Muskhelishvili, *Singular Integral Equations*, Moscow, 1962.
[6] I. N. Vekua, *Soobshch. AN GruzSSR*, 3, No. 9, 869 (1942).
[7] V. V. Ivanov, *Investigations on Modern Problems in the Theory of Functions of a Complex Variable*, Moscow, 1961, p. 439.
[8] I. D. Sofronov, DAN, 110, No. 6, 940 (1956).

[*] It is known that in the case $\varkappa \geq 0$ or $h \equiv 0$, equation (1.1) and problem (2.17) are equivalent ([5,7]).

[**] A number of estimates for $\|K^{-1}\|$ can be obtained from works ([5,8]).

[***] It is known ([1,5]) that a sufficiently broad class of singular equations, both with closed and with open contours of integration, reduces to an equation of the form (1.1).

Note: Figure translations are in progress. See original paper for figures.

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