ELEMENTS OF INFINITE FILTRATION IN \(K\)-THEORY

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 513.836

MATHEMATICS

V. M. BUKHSHTABER, A. S. MISHCHENKO

ELEMENTS OF INFINITE FILTRATION IN K-THEORY

(Presented by Academician P. S. Aleksandrov on 17 III 1967)

By a CW-complex in this paper we mean locally finite CW-complexes in the sense of J. H. C. Whitehead ([1]). In ([2–4]), the functors $\mathcal{K}^*(X)$ and $k^*(X)$ were defined on the category of CW-complexes. M. F. Atiyah and F. Hirzebruch ([3]) proved that, for any compact connected Lie group G, the completed ring of unitary representations $\hat{R}(G)$ is isomorphic to the ring $\mathcal{K}^0(BG)$, and conjectured that $\mathcal{K}^*(BG) = k^*(BG)$, i.e., that the ring $\hat{R}(G)$ is isomorphic to the set of homotopy classes of maps of the space BG into the space BU. In the present paper we prove this conjecture (Theorem 3).

I. In ([9,10]) it was shown that k is a generalized cohomology theory, and that the topology in the ring $k(X)$ generated by the filtration by finite-dimensional skeleta may be non-Hausdorff. In other words, there may exist maps of the complex X into BU that are homotopic to the constant map on each skeleton and are not homotopic to the constant map on the whole complex X. It is also proved there that $\mathcal{K}^*(X) = k^*(X)/\{\bar{0}\}$.

Theorem 1. Let X be a locally finite CW-complex. In order that $\mathcal{K}^0(X) = k^0(X)$, it is necessary and sufficient that for every element $\alpha \in H^{\mathrm{odd}}(X, Q)$ there exist a map $f : SX \to BU$ and a number N such that
$f^*(\operatorname{ch}\eta) = N(s\alpha)+$ terms of higher dimension, where η is the canonical element of $k^0(BU)$. An analogous condition holds for the equality $\mathcal{K}^1(X) = k^1(X)$.

In applications the following formulation is useful:

Theorem $1'$**.** In order that $\mathcal{K}^0(X) = k^0(X)$, it is necessary and sufficient that, in the spectral sequence (see ([3],[4])) converging to $\mathcal{K}^*(X)$ and whose term E_2 is equal to $H^*(X, Z)$, for every element $\alpha \in E_2$ of infinite order there exist a number N such that $N\alpha$ is a cycle of all differentials.

Corollary 1. If $H^{\mathrm{odd}}(X, Q) = 0$, then $\mathcal{K}^0(X) = k^0(X)$; correspondingly, if $H^{\mathrm{ev}}(X, Q) = 0$, then $\mathcal{K}^1(X) = k^1(X)$.

Corollary 2. Let X be a $2l$-connected CW-complex such that the group $H^{2(n+l)}(X, Z)$ has no elements annihilated by the number $(n-1)!$. Then

$\mathcal{K}^0(X) = k^0(X)$. It is easy to formulate the corresponding condition for the equality $\mathcal{K}^1(X) = k^1(X)$.

Corollary 3. Let $\pi : \hat{X} \to X$ be a finite regular covering. $\mathcal{K}^i(\hat{X}) = k^i(X)$ only when $\mathcal{K}^i(X) = k^i(X)$, $i \in Z_2$.

II. Applications.

Theorem 2. Let X be a complex such that $H^*(\Omega X, Q)$ is a finitely generated Q-module. Then $\mathcal{K}^0(X) \approx k^0(X)$.

Proof. According to a well-known theorem of H. Hopf ([5]), under the hypotheses of the theorem the ring $H^*(\Omega X, Q)$ is an exterior algebra on odd-dimensional primitive generators, if X is simply connected. Consequently, in this case the ring $H^*(X, Q)$ is a polynomial algebra on even-dimensional generators. Thus, for the complex X the conditions are satisfied

Corollary 1. Therefore $\mathcal{K}^0(X) = \underleftarrow{\lim}(k^0(X^n)) = k^0(X)$. In the general case one must apply Corollary 3, since, by the hypothesis of the theorem, $\pi_1(X)$ is a finite group.

Theorem 3. *Let G be a compact Lie group. Then the rings $\mathcal{K}^*(BG)$ and $k^*(BG)$ are isomorphic.*

Proof. In view of Corollary 3 it suffices to consider only a connected group G. We note that the group G is homotopy equivalent to the space ΩBG. Hence, applying Theorem 2, we obtain the equality $\mathcal{K}^0(BG) = k^0(BG)$.

We shall now prove that for any element $a \in H^{2i}(BG, Q)$ there exist an element $\eta \in \mathcal{K}^0(BG)$ and a number N such that $\operatorname{ch} \eta = Na+$ terms of higher dimension. Let $\rho : T \subset G$ be a maximal torus. Then we have the commutative diagram

$$\begin{array}{ccc} \mathcal{K}^0(BT) & \overset{\rho^*}{\longleftarrow} & \mathcal{K}^0(BG) \\ \operatorname{ch}\downarrow & & \downarrow\operatorname{ch} \\ H^{**}(BT, Q) & \overset{\rho^{**}}{\longleftarrow} & H^{**}(BG, Q) \end{array}$$

The mapping ρ^{**} is a monomorphism (6). By Corollary 2, for the space $BT = \prod_{i=1}^{r} CP^\infty$ the hypotheses of Theorem 1 are satisfied. Therefore, for any element $a \in H^{2i}(BG, Q)$, there exist such an N and such an element $y \in \mathcal{K}^0(BT)$ that $\operatorname{ch} y = N\rho^{**}a+$ terms of higher dimension. If Γ is the Weyl group of the group G, then the element $z = \sum_{\gamma \in \Gamma} \gamma(y)$ is invariant with respect to the Weyl group, and then, by Theorem II.4.4 of (3), there exists an element $x \in \mathcal{K}^0(BG)$ such that $\rho^*(x) = z$. Then $\operatorname{ch} z = N \cdot \operatorname{ord} \Gamma \cdot a+$ terms of higher dimension. Thus we have proved that the hypotheses of Theorem 1 are satisfied for the space BG.

If the topological group G is not compact, then, generally speaking, $\mathcal{K}^*(BG)$ is not isomorphic to $k^*(BG)$, for example in the case $G = CP^\infty$ (see (9, 10)). However, Theorem 1 implies the following.

Theorem 4. *Let $BO(n, \dots, \infty)$ $(BU(n, \dots, \infty))$ be the n-connected space of the classifying space of the infinite-dimensional orthogonal group O (unitary group U). Then*

$$\mathcal{K}^*(BO(n, \dots, \infty)) = k^*(BO(n, \dots, \infty)), \qquad \mathcal{K}^*(BU(n, \dots, \infty)) = k^*(BU(n, \dots, \infty)).$$

The computation of these rings may be found in (10).

III. **Proof of Theorem 1.** We shall outline the plan of the proof. Let

$$(V)_n \to (V)_{n-1}$$

be a sequence of killing spaces of a CW-complex V and of fibrations with fiber $K(\pi_{n-1}(V),\, n-2)$.

Lemma 1. *Suppose there is given such a sequence of mappings $\{\varphi_n : X \to (V)_n\}$ that*

$$\varphi_n \simeq f_{n+1} \cdot \varphi_{n+1}.$$

Then all the mappings φ_n are homotopic to the constant mapping.

Lemma 2. *Let X and V be CW-complexes such that, for any $i > 0$,*

$$H^i(X, Q) \otimes \pi_{i+1}(V) = 0.$$

If the complex V is an H-space, then the homomorphism

$$\pi : [X, V]_0 \to \lim_{\leftarrow n} [X^n, V]$$

is an isomorphism.

Here $[X, V]_0$ is the group of homotopy classes of mappings with fixed base point, and X^n are the n-dimensional skeleta of the complex X.

Proof. It is enough to prove the equality $\ker \pi = 0$. Let $\varphi : X \to (V)_n$ be a mapping homotopic to the constant mapping on each skeleton X^k. We shall prove that there exists a mapping $\psi : X \to (V)_{n+1}$ such that $f_{n+1} \cdot \psi \simeq \varphi$, and ψ is also homotopic to the constant mapping on each skeleton X^k. Consider the fibration

$$(V)_{n+1} \xrightarrow{K(\pi_n(V),\, n-1)} (V)_n.$$

It is clear that there exists a lifting $\chi : X \to (V)_{n+1}$, i.e. $f_{n+1} \cdot \chi = \varphi$. As follows ...

it is known that in this case every other lifting is uniquely determined by the homotopy class of a map of the space X into the fiber $K(\pi_n(V), n-1)$. In our situation there is a finite number of such homotopy classes. Consequently, among these liftings there is at least one lifting ψ, homotopic to the map into a point on every skeleton X^k. By induction we construct a sequence of maps $\varphi_n : X \to (V)_n$ satisfying the conditions of Lemma 1.

Lemma 3. *If $H^*(X, \mathbf{Z})$ has no torsion, then $\mathcal{K}^*(X) = k^*(X)$.*

It is easy to verify that in the given situation the conditions of Lemma 1 can be made to hold.

It follows from Lemma 3 that for the spaces BU, $BU/[BU]^n$, where $[BU]^n$ is the $2n$-dimensional skeleton of BU,

$$Y = \lim_{\vec{N}} \prod_{i=1}^{N} BU/[BU]^{n_i}, \qquad n_i \to \infty,$$

the groups

$$k^1 = \mathcal{K}^1 = 0.$$

If X satisfies the conditions of Theorem 1, then there exists a map $f : X \to Y$ such that $f : H^{2i}(Y, Q) \to H^{2i}(X, Q)$ is an epimorphism for all i. Then $H^{2i+1}(Y, X; Q) = 0$. By Lemma 2, $\mathcal{K}^0(Y/X) = k^0(Y/X)$. Consider the exact sequence of the pair (Y, X):

$$k^0(Y/X) \leftarrow k^1(X) \leftarrow k^1(Y) = 0.$$

Since the topology in the ring $k^0(Y/X)$ is Hausdorff, we have $k^1(X) = \mathcal{K}^1(X)$ (recall that $k^1(X) = \mathcal{K}^1(X)/\{0\}$). Replacing X by SX, we obtain the isomorphism $\mathcal{K}^0(X) = k^0(X)$.

Now suppose there is an element $a \in H^{2i}(X, Q)$ such that one cannot find $y \in \mathcal{K}^0(X)$ with $\operatorname{ch} y = Na+$ terms of higher dimension. Consider the pair (X, X^{2i}), and for it the commutative diagram:

$$\begin{array}{cccccccccc}
\leftarrow & k^1(X) & \overset{\pi^*}{\longleftarrow} & k^1(X/X^{2i}) & \overset{\delta^*}{\longleftarrow} & k^0(X^{2i}) & \overset{i^*}{\longleftarrow} & k^0(X) & \leftarrow \\
& \downarrow \operatorname{ch} & & & & \downarrow \operatorname{ch} & & \downarrow \operatorname{ch} & \\
\leftarrow & H^{odd}(X/X^{2i}, Q) & \overset{\delta^*}{\longleftarrow} & H^{ev}(X^{2i}, Q) & \overset{i^*}{\longleftarrow} & H^{ev}(X, Q) & \leftarrow & &
\end{array}$$

It is clear that $i^*(a) \neq 0$. Since X^{2i} is a finite complex, there exists an element $y_1 \in \mathcal{K}^0(X^{2i})$ such that $\operatorname{ch} y_1 = N i^* a$. By assumption, $y_1 \notin \operatorname{Im} i^*$. Then $z = \delta^*(y_1) \neq 0$ and $\operatorname{ch} z = 0$. Consider the subgroup $S \subset k^1(X/X^{2i})$ generated by the element z. It can be shown ([10]) that if $\mathcal{K}^1(X) = k^1(X)$, then $\mathcal{K}^1(X/X^{2i}) \simeq k^1(X/X^{2i})$, i.e. $S \cap \{0\} = 0$. Since $\operatorname{ch} z = 0$, the completion $\widehat{S}$ of the group S in the topology of the ring $k^1(X/X^{2i})$ is continual. Hence the set $\operatorname{Im} \delta^* \subset$

$k^1(X/X^{2i})$ is not closed (since $k^0(X^{2i})$ is a finitely generated group). Since $\operatorname{Im}\delta^* = \ker\pi^*$, the set $\{0\} \subset k^1(X)$ is not closed, which was required to be proved.

IV. J. Adams and G. Walker ([8]) constructed an example of a map f of a complex X into a complex Y (Y not locally finite, but finite-dimensional), such that the restriction of f to any skeleton of the space X is homotopic to a map into a point, while f itself is not homotopic to a map into a point. We shall show that such examples also arise in k-theory.

As is known ([7]), for any prime p the group $\pi^p_{2p}(S^3) = \mathbf{Z}_p$, where π^p_k is the p-component of the group π_k. Therefore there exists a complex

$$X_p = S^3 \cup_\varphi (D^{2p+1} \cup_p D^{2p+2}),$$

where $\varphi : S^{2p} \cup_p D^{2p+1} \to S^3$ is the map corresponding to a generator of the group $\pi^p_{2p}(S^3)$, $\widetilde{\varphi} : S^{2p} \to S^3$. The cohomologies $H^*(X_p, \mathbf{Z})$ have the form

$$H^3(X_p, \mathbf{Z}) = \mathbf{Z}$$

(a-generator),

$$H^{2p+2}(X_p, \mathbf{Z}) = \mathbf{Z}_p$$

(b-generator). The integral operation (βP^1) carries the element a into b. Let X be the complex obtained by identifying in all the complexes X_p the spheres S^3 with one another. The resulting complex is locally finite. In an analogous way one may glue a complex using any sequence of primes $\{p_i\}$. Fix some infinite sequence of primes not exhausting

all primes, and denote the corresponding complex by Y. It is easy to see that the complex Y is embedded in X. We compute $k^*(X, \mathbf{Z}_p)$, $\mathcal{K}^*(X)$, $\mathcal{K}^*(Y)$, $\mathcal{K}^*(X/Y)$. For the definition of the functor $k^*(X, \mathbf{Z}_p)$, see (9).

Consider the spectral sequence induced by the filtration by skeleta and, as is easy to show, strongly convergent to $\mathcal{K}^*(X)$. The term

$$E_2^{l,q} = H^l(X, k^q(*)), \qquad q \in \mathbf{Z}_2.$$

All differentials d_r have finite order, and the first nontrivial p-component occurs for the differential d_{2p-1} and is equal to the integral cohomology operation (βP^1) (see (4)). From this it is easy to obtain that

$$E^*_\infty(X) = E^*_\infty(Y) = 0.$$

Thus $\mathcal{K}^*(X) = \mathcal{K}^*(Y) = 0$. The complex X/Y, however, is a bouquet of some p-spheres, and therefore

$$\mathcal{K}^*(X/Y) \neq 0.$$

To compute $k^*(X, \mathbf{Z}_p)$, we apply the analogous spectral sequence, strongly convergent to
$$\mathcal{K}^*(X, \mathbf{Z}_p) = k^*(X, \mathbf{Z}_p). \tag{10}$$

The term
$$E_2^{l,q} = H^l(X, k^q(*, \mathbf{Z}_p)), \qquad q \in \mathbf{Z}_2.$$
Let us write down the nonzero terms:
$$E_2^{3,0} = \mathbf{Z}_p, \qquad E_2^{2p+1,0} = \mathbf{Z}_p, \qquad E_2^{2p+2,0} = \mathbf{Z}_p.$$

The differential
$$d_{2p-1} : E_{2p-1}^{3,0} \to E_{2p-1}^{2p+2,0}$$
is nontrivial; consequently,
$$E_\infty^{2p+1,0} = E_{2p}^{2p+1,0} = \mathbf{Z}_p$$
is the only nontrivial group. Thus
$$k^*(X, \mathbf{Z}_p) = \mathbf{Z}_p.$$

From the computations given above we draw several consequences:

1. For infinite complexes the groups $\mathcal{K}^*(X)$ and $\mathcal{K}^*(X, \mathbf{Z}_p)$ are not related by the Künneth formula.

2. For the $\mathcal{K}^*$-functor there is no exact sequence of a pair; moreover, the closure of the image is, in general, not equal to the kernel. Indeed, the pair (X, Y) constructed above has the sequence
$$\begin{array}{ccccccccc} \leftarrow & \mathcal{K}^*(Y) & \leftarrow & \mathcal{K}^*(X) & \leftarrow & \mathcal{K}^*(X/Y) & \leftarrow & \mathcal{K}^*(Y) & \leftarrow \\ & \| & & \| & & \nparallel & & \| & \\ & 0 & & 0 & & 0 & & 0 & \end{array}$$
which, of course, is not exact.

3. Since $k^*(X, \mathbf{Z}_p) \neq 0$, it follows from the Künneth formula (see (10)) that $k^*(X) \neq 0$; on the other hand $\mathcal{K}^*(X) = 0$, hence
$$k^*(X) = \{\overline{0}\},$$
i.e., all elements of $k^*(X) = [S^{kX}, BU]$ are represented by maps
$$S^{kX} \to BU \qquad (k \geq 0),$$
homotopic to the constant map on each skeleton, but not homotopic to the constant map on the whole complex.

Moscow State University
named after M. V. Lomonosov

Received
17 II 1967

REFERENCES

1. J. H. C. Whitehead, Bull. Am. Math. Soc., **55**, 213 (1949).
2. M. F. Atiyah, Publ. Math. Inst. des Hautes Études, **9**, 23 (1961).
3. M. F. Atiyah, F. Hirzebruch, in: *Collected translations. Mathematics*, **6**, 2, 3 (1962).
4. L. Hodgkin, On the K-Theory of Lie Groups, Coventry, 1966.
5. J.-P. Serre, in: *Collected translated works*, Moscow, 1958, p. 9.
6. A. Borel, ibid., p. 161.
7. J.-P. Serre, ibid., p. 124.
8. J. Adams, G. Walker, *Collected translations. Mathematics*, **9**, 1 (1965).
9. V. M. Bukhshtaber, A. S. Mishchenko, Abstracts of reports of the Fifth All-Union Topological Conference, Novosibirsk, 1967, p. 7.
10. V. M. Bukhshtaber, A. S. Mishchenko, Izv. Acad. Sci. USSR, Ser. Math., **32**, No. 1 (1968).

Note: Figure translations are in progress. See original paper for figures.