ON THE CAUCHY PROBLEM FOR ONE CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER IN HILBERT SPACE

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Abstract

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ON THE CAUCHY PROBLEM FOR ONE CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER IN HILBERT SPACE

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1. In the present paper we study the Cauchy problem

$$d^2u/dt^2 + A(t)u(t) + M(t, u(t), du(t)/dt) = 0; \tag{1}$$

$$0 \leq t < +\infty, \qquad u(0) = \varphi, \qquad u'(0) = \psi, \tag{2}$$

where $A(t)$, for each fixed t, is a linear operator in the Hilbert space H, whose boundedness is not assumed. The function $M(t, u, v)$ (not necessarily linear) has values in H.

The problem (1), (2) in the case of a constant operator A and a function $M = M(u)$, independent of t and du/dt, was studied by Browder in (^{1}). He established an existence and uniqueness theorem for the solution on the half-axis. In the case of a twice continuously differentiable operator $A(t)$ and a function $M = M(u)$, the problem (1), (2) was studied in (^{2}), where the existence of a weak solution was in fact established. Existence and uniqueness theorems for the solution of the Cauchy problem (1), (2) were obtained by P. E. Sobolevskii and V. A. Pogorelenko (^{3,8}) under other assumptions on the function $M(t, u, v)$.

In the present paper assumptions are established concerning the existence, uniqueness, and continuous dependence on the initial data of the solution of the problem (1), (2) on the half-axis. In doing so, ideas from Browder' s work (^{1}) are used. In the last section of the paper the results obtained are applied to the study of a mixed boundary-value problem for hyperbolic equations of order $2m$ with weak nonlinearity.

2. We shall assume that:

I. $A(t)$, $0 \leq t < +\infty$, is a family of self-adjoint operators in the Hilbert space H, and

$$(A(t)u, u) \geq k(t)(u, u), \tag{3}$$

where $k(t)$ is a positive continuous function. The domain of definition of $A(t)$ does not depend on t, and $A(t)A^{-1}(0)$ is strongly continuously differentiable with respect to t.

II. The function $M(t, u, v)$, acting from $[0, +\infty) \times D(A^{1/2}) \times H$ into H, satisfies the following conditions:

For any $T > 0$, $C > 0$ there exists a constant $K(C, T)$ such that:

1) if $|t| \leq T$, $|t_1| \leq T$, $\|A^{1/2}u\| \leq C$, $\|v\| \leq C$, $\|A^{1/2}u_1\| \leq C$, $\|v_1\| < C$, then

$$\|M(t, u, v)\| \leq K(C, T); \qquad \|M(t, u, v) - M(t_1, u_1, v_1)\| \leq$$

$$\leq K(C, T) \left\{|t - t_1| + \|A^{1/2}(u - u_1)\| + \|v - v_1\|\right\};$$

2) if $u(t)$ is a strongly continuously differentiable function from $[0, T]$ into $D(A^{1/2})$ and $\|A^{1/2}(0)u(0)\| + \|u'(0)\| \leq C$, then

$$\operatorname{Re} \int_0^t \left(M\left(s, u(s), \frac{du(s)}{ds}\right), \frac{du(s)}{ds} \right) ds \geq$$

$$\geq -K(C, T) \left\{ 1 + \int_0^t \left[\|A^{1/2}(s)u(s)\|^2 + \left\|\frac{du}{ds}\right\|^2 \right] ds \right\}$$

for all $0 \leq t \leq T$;

3) for any strongly continuously differentiable $u : [0, T] \to D(A^{1/2})$, $v : [0, T] \to H$ such that $\|A^{1/2}(t)u(t)\| + \|v(t)\| \leq C$, $M(t, u, v)$ is strongly continuously differentiable with respect to t, and

$$\left\|\frac{d}{dt} M(t, u(t), v(t))\right\| \leq K(C, T) \left\{ \left\|A^{1/2}(t)\frac{du}{dt}\right\| + \left\|\frac{dv}{dt}\right\| + 1 \right\}$$

for all $t \in [0, T]$;

4) for any strongly continuously differentiable $u, u_1 : [0, T] \to D(A^{1/2})$; $v, v_1 : [0, T] \to H$ such that $u, u_1 \in D(A)$; du/dt, $du_1/dt \in D(A^{1/2})$; $v, v_1 \in D(A^{1/2})$ and

$$\left\|A^{1/2}(t)\frac{du}{dt}\right\| + \|A(t)u(t)\| + \|A^{1/2}(t)v(t)\| + \left\|\frac{dv}{dt}\right\| \leq C;$$

$$\left\|A^{1/2}\frac{du_1}{dt}\right\| + \|Au_1(t)\| + \|A^{1/2}v_1(t)\| + \left\|\frac{dv_1}{dt}\right\| \le C,$$

the inequality

$$\left\|\frac{d}{dt}\{M(t,u(t),v(t)) - M(t,u_1(t),v_1(t))\}\right\| \le$$

$$\le K(C,T)\left\{\|A(t)u(t) - A(t)u_1(t)\| + \left\|A^{1/2}(t)\frac{du}{dt} - A^{1/2}(t)\frac{du_1}{dt}\right\| + \right.$$

$$\left. + \|A^{1/2}(t)v(t) - A^{1/2}(t)v_1(t)\| + \left\|\frac{dv_1}{dt} - \frac{dv}{dt}\right\|\right\}$$

holds for all $t \in [0,T]$.

As is known, it follows from condition I that $A^{1/2}(t)$ has a constant domain of definition (see [4]). We note that condition I is equivalent to the following condition (see [5]): $A(t)$ is a family of self-adjoint operators satisfying inequality (3), the operator $A^{-1}(t)$ is strongly continuously differentiable, and $A(t)\,dA^{-1}(t)/dt$ is strongly continuous.

3. Starting from these conditions and using Kato' s results (see [6]), by the method of semigroup theory we establish various assertions on the existence and properties of the exact solution of the problem posed.

By an exact solution of the Cauchy problem (1), (2) we mean a function $u(t)$ satisfying the following conditions: $u(t)$ is twice strongly continuously differentiable, $u(t) \in D(A)$, $du/dt \in D(A^{1/2})$, for all $t \in [0,+\infty]$; $A(t)u(t)$, d^2u/dt^2, $A^{1/2}u$, and $A^{1/2}(t)\,du/dt$ are continuous as functions of t on $[0,+\infty)$ with values in H, and $u(t)$ satisfies equation (1) and the initial data (2).

Theorem 1. *Let conditions I and II be fulfilled. Then the Cauchy problem (1), (2) has an exact solution* $u(t)$ *on the half-axis* $[0,+\infty)$ *for* $\varphi \in D(A)$ *and* $\psi \in D(A^{1/2})$.

Theorem 2. *Let condition I and 1), 2), 3) of II be fulfilled. Then for any* $T > 0$, $C > 0$, *there exists a constant* $K(C,T)$ *such that, for the solutions* $u(t)$, $u_1(t)$ *of the Cauchy problem (1), (2) with corresponding initial data* $\varphi, \psi, \varphi_1, \psi_1$, *where* $\|A^{1/2}(0)\varphi\| + \|\psi\| \le C$, $\|A^{1/2}(0)\varphi_1\| + \|\psi_1\| \le C$, *the inequality*

$$\|A^{1/2}(t)u(t) - A^{1/2}(t)u_1(t)\|^2 + \|u'(t) - u_1'(t)\|^2 \le$$

$$\le K(C,T)\{\|A^{1/2}\varphi - A^{1/2}\varphi_1\|^2 + \|\psi - \psi_1\|^2\}$$

holds for all $t \in [0,T]$.

It follows from Theorem 2 that the Cauchy problem (1), (2) is unique on $[0, +\infty)$.

Theorem 3. *Suppose that conditions I and II are satisfied. Then for any* $T > 0$, $C > 0$ *there exists a constant* $K(C, T)$ *such that for solutions* $u(t)$, $u_1(t)$ *of the Cauchy problem (1), (2) with corresponding initial data* $\varphi, \psi, \varphi_1, \psi_1$ *satisfying the relations*

$$\|A(0)\varphi\|^2 + \|A^{1/2}(0)\varphi\|^2 + \|\psi\|^2 + \|A^{1/2}(0)\psi\|^2 \leq C;$$

$$\|A^{1/2}(0)\varphi_1\|^2 + \|A(0)\varphi_1\|^2 + \|\psi_1\|^2 + \|A^{1/2}(0)\psi_1\|^2 \leq C$$

the inequality

$$\|u''(t) - u_1''(t)\|^2 + \|A^{1/2}(t)u'(t) - A^{1/2}(t)u_1'(t)\|^2 \leq$$

$$\leq K(C, T)\{\|A^{1/2}(0)\varphi - A^{1/2}(0)\varphi_1\|^2 + \|A(0)\varphi - A(0)\varphi_1\|^2 +$$

$$+\|A^{1/2}(0)\psi - A^{1/2}(0)\psi_1\|^2 + \|\psi - \psi_1\|^2\}$$

holds for all $t \in [0, T]$.

4. Example. In the Hilbert space $L_2(G)$, where G is a bounded domain with boundary Γ, consider the equation

$$\frac{\partial^2 u}{\partial t^2} + (-1)^m \sum_{|\alpha| \leq 2m} a_\alpha(x, t) D^\alpha u = f\left(t, x, u, D^\beta u, \frac{\partial u}{\partial t}\right) \qquad (|\beta| \leq m) \qquad (4)$$

with initial conditions

$$u(0, x) = \varphi(x), \qquad \frac{\partial u}{\partial t}(0, x) = \psi(x) \qquad (5)$$

and boundary conditions

$$u|_\Gamma = 0, \qquad D^\beta u|_\Gamma = 0 \qquad (|\beta| \leq m). \qquad (6)$$

Here $\alpha = (\alpha_1, \dots, \alpha_n)$; $x = (x_1, \dots, x_n)$;

$$D^\alpha = \partial^{\alpha_1 + \dots + \alpha_n} / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n};$$

$|\alpha| = \alpha_1 + \dots + \alpha_n$.

We assume that:

a) the boundary Γ belongs to the class C^{2m+p} $(p \geq m)$, $a_\alpha(x, t) \in C_x^{|\alpha|+p}((G \cup \Gamma) \times R^+)$, where $R^+ = [0, +\infty)$, and is continuously differentiable with respect to t in $[G \cup \Gamma] \times R^+$,

$$\sum_{|\alpha| \leq 2m} a_\alpha(x, t) D^\alpha$$

is a strongly elliptic expression with real coefficients of order $2m$ for each t, i.e. for any $t \geq 0$, uniformly in x:

$$(-1)^m \sum_{|\alpha|=2m} a_\alpha(x,t)\xi^\alpha \geq \varepsilon(t)|\xi|^{2m},$$

where $\varepsilon(t) > 0$, $x \in G \cup \Gamma$, the real vector $\xi = (\xi_1, \dots, \xi_n) \in E^n$, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$, $|\xi|^2 = \xi_1^2 + \dots + \xi_n^2$, and $2m > n$;

b) $f(t, x, u, v_\beta, w)$ is a numerical function on

$$\Delta = R^+ \times [G \cup \Gamma] \times R \times \prod_{0 \leq |\beta| \leq m} R_\beta \times R,$$

continuously differentiable in all arguments, and all first-order partial derivatives are bounded in Δ and satisfy the Lipschitz condition with respect to u, v_β, w in every bounded subdomain Δ;

c) the expression

$$\sum_{|\alpha| \leq 2m} a_\alpha(x,t) D^\alpha$$

is formally self-adjoint for each t, i.e.

$$\sum_{|\alpha| \leq 2m} (-1)^{|\alpha|} D^\alpha (a_\alpha(x,t)u) = \sum_{|\alpha| \leq 2m} a_\alpha(x,t) D^\alpha u \quad \text{for } u \in C^{2m}(G).$$

Define the operator $L(t)$:

$$D(L(t)) = \mathring{\dot{W}}_2^{2m} \cap \mathring{\dot{W}}_2^{m}, \qquad L(t)u = \sum_{|\alpha| \leq 2m} a_\alpha(x,t) D^\alpha u \quad \text{for } u \in D(L(t)).$$

The operator $L(t)$ is self-adjoint (see [7]).

We shall call a generalized solution of problem (4), (5), (6) a function $u(t,x)$ defining a twice strongly continuously differentiable mapping $u(t)$ from $[0, +\infty)$ into $\mathcal{L}_2(G)$ such that, for each t, $u(t) \in W_2^{2m} \cap \dot{W}_2^m$, $du/dt \in \dot{W}_2^m$; $u(t)$ and du/dt are continuous as functions from $[0, +\infty)$ into $W_2^{2m} \cap \dot{W}_2^m$ and $\dot{W}_2^m$, respectively, $u(0) = \varphi \in W_2^{2m} \cap \dot{W}_2^m$, $du/dt|_{t=0} = \psi(x) \in \dot{W}_2^m$, and $u(t)$ satisfies the equation

$$d^2u/dt^2 + L(t)u = f\left(t, x, u, D^\beta u, \frac{\partial u}{\partial t}\right).$$

Theorem 4. *Suppose that conditions a), b), c) are fulfilled. Then the following assertions hold:*

1) *For any $\varphi(x) \in W_2^{2m} \cap \dot{W}_2^m$, $\psi(x) \in \dot{W}_2^m$, there exists a unique generalized solution of problem (4), (5), (6) on $[0, +\infty)$.*

2) *If* $u(t,x)$, $u_1(t,x)$ *are solutions of problem (4), (5), (6) with corresponding initial data* φ, ψ *and* φ_1, ψ_1, *where* $\|\varphi\|_{W_2^m(G)} + \|\psi\|_{\mathcal{L}2(G)} \le C$, $\|\varphi_1\|_{W_2^m(G)} + \|\psi_1\|_{\mathcal{L}2(G)} \le C$, *then for any* $T > 0$, $C > 0$, *there exists a constant* $K(T,C)$ *such that the inequality*

$$\|u(t,x) - u_1(t,x)\|^2_{W_2^m(G)} + \left\|\frac{\partial u(t,x)}{\partial t} - \frac{\partial u_1(t,x)}{\partial t}\right\|^2_{\mathcal{L}2(G)} \le$$

$$\le K(T,C)\{\|\varphi - \varphi_1\|^2_{W_2^m} + \|\psi - \psi_1\|^2_{\mathcal{L}2}\}$$

holds for every $t \in [0,T]$.

3) *If* $u(t,x)$, $u_1(t,x)$ *are solutions of problem (4), (5), (6) with corresponding initial data* φ, ψ *and* φ_1, ψ_1, *satisfying the relations:*

$$\|\varphi\|_{W_2^{2m}(G)} + \|\psi\|_{W_2^m(G)} \le C; \qquad \|\varphi_1\|_{W_2^{2m}(G)} + \|\psi_1\|_{W_2^m(G)} \le C,$$

then for any $T > 0$, $C > 0$, *there exists a constant* $K(T,C)$ *such that the inequality*

$$\left\|\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u_1(t,x)}{\partial t^2}\right\|^2_{\mathcal{L}2(G)} + \left\|\frac{\partial u(t,x)}{\partial t} - \frac{\partial u_1(t,x)}{\partial t}\right\|^2_{W_2^m(G)} \le$$

$$\le K(T,C)\{\|\varphi - \varphi_1\|^2_{W_2^{2m}(G)} + \|\psi - \psi_1\|^2_{W_2^m(G)}\}$$

holds for all $t \in [0,T]$.

Note added in proof. A related problem was considered by S. Ya. Yakubov [9]; this work became known to the author after the present article had been written.

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Note: Figure translations are in progress. See original paper for figures.

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