THE THEORY OF MODULAR FORMS AND THE PROBLEM OF FINDING FORMULAS FOR THE NUMBER OF REPRESENTATIONS OF NUMBERS BY POSITIVE QUADRATIC FORMS

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Abstract

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THE THEORY OF MODULAR FORMS AND THE PROBLEM OF FINDING FORMULAS FOR THE NUMBER OF REPRESENTATIONS OF NUMBERS BY POSITIVE QUADRATIC FORMS

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In this note a method is developed which makes it possible, on the basis of the theory of modular forms and the investigations of I. M. Vinogradov and Burgess on the least quadratic nonresidue, to solve the question of the existence of Liouville-type formulas for the number of representations of numbers by positive quadratic forms. In the case where such a formula exists (so far only for quadratic forms in 6 variables), an algorithm is constructed for finding it. In addition, on the basis of Eichler' s results on the representability of certain modular forms in the form of generalized quaternary theta series, a method is developed for constructing Buligyn-Mordell type formulas for the number of representations of numbers by quadratic forms in eight variables.

Let

$$Q(x_1, \dots, x_{2k}) \tag{1}$$

be an integral positive quadratic form in $2k$ variables; $k \geq 2$ is an integer. We represent the theta series corresponding to it as a sum of two terms:

$$\sum_{x_1,\dots,x_{2k}=-\infty}^{\infty} \exp[2\pi i\tau Q(x_1, \dots, x_{2k})] = E(\tau) + \theta(\tau). \tag{2}$$

Here τ is a complex number, $\operatorname{Im}\tau > 0$; $E(\tau)$ is the Eisenstein series corresponding to the theta series appearing on the left-hand side of (2); $\theta(\tau)$ is a cusp form. If the form $\theta(\tau)$ can be represented as a finite linear combination of generalized binary theta series of the form

$$\sum_{x_1,x_2=-\infty}^{\infty} P_{k-1}(x_1,x_2)\exp\left[2\pi it\frac{G(x_1,x_2)}{N^2}\tau\right], \qquad x_r \equiv h_r \pmod N, \quad r=1,2, \tag{3}$$

where $t > 0$ is an integer; $G(x_1, x_2)$ is an integral positive quadratic form of level N; $P_{k-1}(x_1, x_2)$ is a spherical function of order $(k-1)$ with respect to G; h_1 and h_2 are integers satisfying

$$\frac{\partial}{\partial h_r}G(h_1,h_2) \equiv 0 \pmod N,$$

then the quadratic form $Q = Q(x_1, \dots, x_{2k})$ will be called **Liouvillean**. Moreover, if one may restrict oneself to series (3) with $h_1 = h_2 = 0$, $t = 1$, then Q is called a **Liouvillean form in the narrow sense**.

Comparison of coefficients in formula (2) leads to formulas for the number $M[n = Q]$ of representations of the number n by the form Q. From the fact that the form Q is Liouvillean it follows that for it there is an asymptotic formula with a remainder term of the form $B_\varepsilon n^{(k-1)/2+\varepsilon}$. Such results in the general case can so far be obtained only on the basis of Peterson' s hypothesis on the eigenvalues of Hecke operators, proved

by Eichler ([5]) only for the case $k = 2$. Apparently, there is only a finite number of classes of primitive Liouville quadratic forms. Below we substantiate some particular cases of this proposition.

Liouville and later a number of other authors (see ([4])) found formulas for the number of representations of numbers by many concrete quadratic forms in 4 and 6 variables. In these formulas the principal terms depend on the divisors of the number being represented, while the supplementary terms are sums of values of certain simple functions taken over representations of numbers by binary positive quadratic forms. It can be shown that all quadratic forms for which these formulas were found are Liouville forms; moreover, all quadratic forms in 6 variables turn out to be Liouville in the narrow sense.

Using some considerations from Rankin' s work ([8]), results from the theory of modular forms ([7,9]), and applying Burgess's estimate ([3]) for the least quadratic nonresidue modulo a prime modulus, we obtain the following proposition.

Theorem 1. For every real number $\varepsilon > 0$ there exists a number C_0, depending only on ε, such that if the prime number q and the integer $l \geq 0$ satisfy the conditions

$$q > C_0, \qquad (k-1)\left(\frac{1}{4\sqrt{e}}+\varepsilon\right)-\frac{1}{2} < l \leq k-1, \tag{4}$$

then none of the integral positive quadratic forms of type $(-k, q, \chi)$ $\left(\chi(n) = \left(\frac{n}{q}\right) \text{ is the Jacobi symbol, } k \geq 3 \text{ odd}\right)$ and discriminant $-q^{2l+1}$ is Liouville in the narrow sense.

In particular, for $k = 3$ (for forms in 6 variables) the second condition (4) is always fulfilled (for sufficiently small ε). The same can be said for any odd $k > 3$, if one assumes the validity of the well-known hypothesis of I. M. Vinogradov ([1], [2]) on the least quadratic nonresidue modulo a prime modulus q (the least nonresidue $(\bmod q)$ is a quantity of order q^{ε}).

Formulas of Liouville type in the narrow sense for positive quadratic forms in 6 variables can be constructed with the aid of the results of ([7], [9]) and the following proposition.

Theorem 2. Let $Q = Q(x) = Q(x_1, \dots, x_k)$ and $R = R(x) = R(x_1, \dots, x_k)$ be integral positive quadratic forms with an even number k of variables, and let the form $Q(x)$ be equivalent to the form $R(x)$. Let the linearly independent quadratic polynomials $W_1(x), \dots, W_{k(k+1)/2-1}(x)$ form a basis of the set of spherical functions of the second order with respect to $Q(x)$, and let the linearly independent quadratic polynomials $V_1(x), \dots, V_{k(k+1)/2-1}(x)$ form a basis of the set of spherical functions of the second order with respect to $R(x)$. Then each series

$$\sum_x V_m(x) z^{R(x)} \left(m = 1, \dots, \frac{k(k+1)}{2} - 1\right), \qquad z = \exp(2\pi i \tau),$$

is a finite linear combination of the series

$$\sum_x W_t(x) z^{Q(x)} \left(t = 1, \dots, \frac{k(k+1)}{2} - 1\right).$$

This theorem, in particular, makes it possible, for every integral positive quadratic form $Q = Q(x) = Q(x_1, \dots, x_8)$ of type $(-4, N, 1)$, of squarefree level $N > 1$, to construct a formula for $M[n = Q]$, under the condition $(n, N) = 1$, of the Bulygin–Mordell type, i.e. a formula whose supplementary term is a sum of values of certain simple functions taken over representations of numbers by positive quaternary

quadratic forms. The construction of a formula for $M[n = Q]$ can be carried out in the following way. Let N_ν $(\nu = 1, \dots, \tau(N) - 1)$ run through all divisors of the number N greater than 1. There is a finite number of classes of integral quaternary positive quadratic forms of type $(-2, N_\nu, 1)$. From each class choose one representative:

$$G_{\nu_1}(x_1, \dots, x_4), \dots, G_{\nu k_\nu}(x_1, \dots, x_4) \quad (\nu = 1, \dots, \tau(N) - 1).$$

We seek the formula for $M[n = Q]$ in the form

$$M[n = Q] = \rho(n) + \sum_{\nu=1}^{\tau(N)-1} \sum_{i=1}^{k_\nu} \sum_{1\le r\le s\le 4} c_{rs}^{(\nu i)} \sum_{n=G_{\nu i}(x_1,\ldots,x_4)} \{x_r x_s - g_{rs}^{(\nu i)}\}, \qquad (5)$$

where $\rho(n)$ is the coefficient of z^n $(z = \exp(2\pi i t))$ in the expansion of the Eisenstein series corresponding to the form Q,

$$g_{rs}^{(\nu i)} = \frac{n}{2}(2G_{\nu i})_{rs}^{-1};$$

here $(2G_{\nu i})^{-1}$ denotes the (r, s)-th element of the matrix inverse to the matrix of the form $2G_{\nu i}$. The undetermined coefficients $c_{rs}^{(\nu i)}$ can be determined from the system of linear equations obtained if in formula (5) one assigns to the number n a certain number of concrete values. From the results of Eichler (4), Hecke (7), Schoeneberg (9), and Theorem 2 it follows that, in our case, this procedure always leads to a formula of the indicated type. In particular, formulas for some quadratic forms in 8 variables of degree 11 were obtained in this way.

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Note: Figure translations are in progress. See original paper for figures.