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Abstract

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ON THE RIGIDITY OF DARBOUX SURFACES WITH BOUNDARY IN A RIEMANNIAN SPACE

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The paper establishes conditions for the rigidity of Darboux surfaces with boundary in a Riemannian space, subject on the boundary to a generalized sliding condition. As a special case, one obtains rigidity conditions for Darboux surfaces with boundary under nonorthogonal sleeve constraints.

Let us give a precise formulation of the result. Consider in a Riemannian space R_3 a Darboux surface S of positive extrinsic curvature $K \geq k_0 > 0$, with boundary L, $L \in C^{2,\alpha}$, $0 < \alpha < 1$. Denote by E_3 the Euclidean space tangent to R_3 along the curve L. In E_3 we shall consider along L the vectors $\mathbf{t}, \mathbf{n}, ...$, regarding them as objects—tensors—of the space R_3. Let $\mathbf{t}$ be the vector tangent to L; $\mathbf{n}$ the normal to the strip S along L; $\eta = [\mathbf{tn}]$. Consider along L the moving frame $\{\mathbf{t}, \mathbf{n}, \eta\}$ and a certain field $\mathbf{l} = \mathbf{l}(s)$ of class $C^{1,\alpha}$, $0 < \alpha < 1$. Denote the angle between $\mathbf{n}$ and $\mathbf{l}$ by α, taking the measurement from $\mathbf{n}$ to $\mathbf{l}$ counterclockwise when viewed from the side of the vector $\mathbf{t}$. Let $\mathbf{l}_\tau$ be the projection of $\mathbf{l}$ onto the tangent plane to S, and let $\beta = \beta(s)$ be the angle between η and $\mathbf{l}_\tau$, where the measurement is made in the positive direction when viewed from the side of the vector $\mathbf{n}$. In this notation the vector field $\mathbf{l}$ is uniquely determined by specifying the angles $\alpha = \alpha(s)$ and $\beta = \beta(s)$ as functions of arc length of the contour L.

We shall study infinitesimal bendings of the surface S, assuming that some interior point of the surface is fixed together with the tangent plane, and that along the boundary L the bending field $\mathbf{z}(s)$ satisfies the condition $\mathbf{z}(s)\mathbf{l}(s) = 0$, where $\mathbf{l}(s)$ is the given vector field. We shall call these conditions the condition of generalized sliding generated by the vector field $\mathbf{l}$. If $\beta(s) \equiv 0$ for $\mathbf{l}(s)$, then these conditions are called a sleeve constraint.

Theorem 1. *Suppose that along L there is given a family of vector fields $\mathbf{l}_\alpha(s)$, determined by angles $\alpha(s)$ and $\beta_0(s)$, where $\beta_0(s)$ is a fixed function from the interval $(-\pi/2, \pi/2)$, while $\alpha(s)$ is arbitrary. Subject the Darboux surface to the condition of generalized sliding generated by the vector field $\mathbf{l}_\alpha$ for fixed $\alpha(s)$. Then there exists a constant $\alpha_0 > 0$ such that the Darboux surface with the condition $\mathbf{zl} = 0$ along L is rigid if $\pi/2 - \alpha_0 < \alpha(s) < \pi/2$. The constant α_0 is determined by the surface and the prescribed function $\beta_0(s)$.*

Theorem 2. *There exists a constant* $\alpha_0 > 0$*, depending on the surface, such that any sleeve constraint of the Darboux surface, determined by an angle* $\alpha(s)$*,* $\pi/2 - \alpha_0 < \alpha(s) < \pi/2$*, ensures the rigidity of the surface* S*.*

Theorem 2 is a special case of Theorem 1 when $\beta_0(s) \equiv 0$.

We shall divide the proof of Theorem 1 into several stages.

1°. Derivation of the bending equations. Introduce on S an isothermally conjugate parametrization mapping S onto the unit disk D of the plane (u^1, u^2) with boundary Γ. Let $a_{\alpha\beta}du^\alpha du^\beta$, $b_{\alpha\beta}du^\alpha du^\beta$ be the first and second fundamental forms of the surface; then $a = |a_{\alpha\beta}| \neq 0$, $b_{\alpha\beta} = b_0\delta_{\alpha\beta}$, $b \neq 0$, $\delta_{\alpha\beta}$ is the Kronecker symbol. In the space R_3 construct a semigeodesic parametrization, taking the surface S as the base. Then the equations of infinitesimal bendings of the surface S are written in the form $z_{(\alpha,\beta)} = b_{\alpha\beta}z_3$, where z_i is the covariant tensor of displacement of points of the surface. Introducing into consideration the complex function $w(z) = z_1 + iz_2$, $z =$

$= u^1 + iu^2,\ i^2 = -1$, we write these equations in the form $(*)$ $\partial_{\bar z} w + Aw + B\bar w = 0,\ z \in D$, where $A = -\partial_{\bar z} \ln \sqrt{a}\sqrt{K}$; $B = \dfrac{1}{2b_0}(\theta_{111} + i\theta_{112}) = -\dfrac{1}{2b}(\theta_{221} + i\theta_{222})$; here θ_{ijh} is the Darboux tensor of the surface S. Since on Darboux surfaces $\theta_{ijh} \equiv 0$, on them the bending equations can be written in the form

$$\partial_{\bar z}\varphi = 0, \tag{1}$$

where $\varphi = w/\sqrt{a}\sqrt{K}$. By assumption, some interior point of the Darboux surface is fixed together with the tangent plane at it. Without loss of generality, one may take this point to be $z = 0$, and therefore the desired function $\varphi(z)$ must have a zero of second order at the point $z = 0$, i.e.

$$\varphi(z) = z^2\varphi_1(z), \quad \text{where } \varphi_1(z) \text{ is holomorphic in } D. \tag{2}$$

Equation (1) with condition (2) will be regarded as the equation of infinitesimal bendings of Darboux surfaces.

p. 2°. **Derivation of the boundary conditions.** From the notation introduced above it follows that

$$\mathbf{l} = \mathbf{n}\cos\alpha + (\mathbf{t}\sin\beta_0 + \eta\cos\beta_0)\sin\alpha,$$

and therefore the sliding condition takes the form

$$(zn)\cos\alpha + (zt)\sin\beta_0\sin\alpha + (z\eta)\cos\beta_0\sin\alpha = 0.$$

We write this condition in coordinate form, introducing into consideration the coordinates η^i and t^i $(i = 1, 2, 3)$ of the vectors η and $\mathbf{t}$. Noting that $\eta^3 = t^3 = 0$, we have

$$z_\alpha(t^\alpha \sin\beta_0 + \eta^\alpha \cos\beta_0)\sin\alpha + z_3\cos\alpha = 0. \tag{3}$$

Since, by assumption, $L \in C^{2,\alpha},\ 0 < \alpha < 1$, it follows that $t^\alpha, \eta^\alpha \in C^{1,\alpha},\ 0 < \alpha < 1$. Denote by l^i the coordinates of the vector $\mathbf{l}$; then $l^1 = t^1\sin\beta_0 + \eta^1\cos\beta_0$; $l^2 = t^2\sin\beta_0 + \eta^2\cos\beta_0$. Since $\beta_0(s) \in (-\pi/2, \pi/2)$, we have $\operatorname{Ind}(l^1 + il^2) = +1$. We also note that the vector $\{l^1, l^2, 0\} \equiv \mathbf{l}_\tau$ is not tangent to L at any point, and therefore, in view of the topological mapping of S onto the plane (u^1, u^2), at no point of Γ is the direction $\{l^1, l^2\}$ tangent to Γ.

We use the relation

$$z_3 = \frac{1}{2b_0}(z_{1,1} + z_{2,2}) \equiv \frac{1}{b_0}\operatorname{Re}\{\partial_z w + \partial_z \ln\sqrt{K}\, w\}$$

and rewrite condition (3) in complex form. We have

$$\operatorname{Re}\{\partial_t w(t) + \partial_t \ln\sqrt{K}\, w(t) + \operatorname{tg}\alpha\cdot\overline{\lambda(t)}\, w(t)\} = 0, \quad t \in \Gamma,$$

where $\lambda(t) = b_0(l^1 + il^2)$. Passing in this relation to the function $\varphi(z)$, we obtain the desired boundary condition:

$$\operatorname{Re}\{\partial_t\varphi(t) + \partial_t \ln\sqrt{aK}\sqrt{K}\,\varphi(t) + \operatorname{tg}\alpha\cdot\overline{\lambda(t)}\,\varphi(t)\} = 0. \tag{4}$$

p. 3°. **Auxiliary problem.** We solve the problem: find in D a holomorphic function $\chi(z)$, continuous in the closed domain, vanishing at the origin, nonzero on the boundary, and satisfying there the boundary condition

$$\operatorname{Re}\{i\lambda(t)\chi(t)\} = 0. \tag{5}$$

We shall show that this problem is solvable. Let $\chi(z) = z\chi_1(z)$, where $\chi_1(z)$ is the sought holomorphic function in D. Then $\chi_1(z)$ satisfies on Γ the condition

$$\operatorname{Re}\{i\lambda(t)t\chi_1(z)\} = 0.$$

Since $\operatorname{Ind}\{i\lambda(t)t\} = 0$, the obtained boundary-value problem has a solution $\chi_1^*(z)$ (^{1}), which vanishes nowhere either in the domain or on its boundary. The function $\chi_1^*(z)$ can be represented in the form

$$\chi_1^*(z) = i\beta_0 e^{i\gamma(z)}, \quad \text{where } \beta_0 \neq 0,\ \operatorname{Im}\beta_0 = 0,$$

$$\gamma(z) = \frac{1}{\pi i}\int_\Gamma \frac{\arg\lambda(\tau)\,\vec{\tau}}{\tau - z}\,d\tau + \frac{1}{2\pi i}\int_\Gamma \arg\lambda(\tau)\,\overrightarrow{d\tau}.$$

We take as the desired function $\chi(z)$ the function $\chi(z) = z\chi_1^*(z)$, which will indeed be a solution of the posed problem. We shall show that the function

$\chi(z)$ has a derivative $\partial_z \chi$ continuous in the closed domain. For this it is enough to show that $\partial_z \gamma$ is continuous in $D + \Gamma$.

Using the known rules for differentiating an integral of Cauchy type, we find

$$\partial_z \gamma(z) = \frac{1}{\pi i} \int_\Gamma \frac{\partial}{\partial \tau} (\arg \lambda(\tau)\, \overline{\tau}) \frac{d\tau}{\tau - z}.$$

Since $\arg \lambda(t) t \in C^{1,\alpha}$, we have

$$\frac{\partial}{\partial t} (\arg \lambda(t) t) \in C^{0,\alpha}, \quad 0 < \alpha < 1,$$

and therefore $\partial_z \gamma(z)$ is continuous in the closed domain.

Let us now establish that the direction $\{\chi_1, \chi_2\}$, $\chi_1 + i\chi_2 = \chi(t)$, $t \in \Gamma$, is nowhere tangent to Γ at the boundary. Indeed, by (5) we have

$$\mathrm{Re}\{\, i\overline{(l^1 + il^2)}(\chi_1 + i\chi_2)\,\} = -l^2 \chi_1 + l^1 \chi_2 = 0.$$

This relation indicates that the vectors $\{\chi_1, \chi_2\}$ and $\{-l^2, l^1\}$ are orthogonal in the plane (u^1, u^2), and hence the vector $\{\chi_1, \chi_2\}$ coincides with the direction of the vector $\{l^1, l^2\}$, which, as was already noted above, is nowhere tangent to Γ. We also note that, since $\chi(t) \neq 0$, $\lambda(t) \neq 0$ on Γ, the function $\mathrm{Re}\{\overline{\lambda(t)}\chi(t)\}$ never vanishes.

4°. On the solvability of problem (1), (2), (4). We shall show that there exists an a_0, $a_0 > 0$, depending on the surface and on the function $\beta_0(s)$, such that every solution of problem (1), (2), (4) is identically equal to zero for all functions $\alpha(s)$ satisfying the inequality

$$\pi/2 - a_0 < \alpha(s) < \pi/2.$$

Suppose the contrary: that for every arbitrarily small $\varepsilon > 0$ there is an angle $\tilde{\alpha}(s)$ from the interval $(+\pi/2 - \varepsilon, \pi/2)$ such that the boundary-value problem (1), (2), (4) corresponding to this angle has a nonzero solution $\varphi = \widetilde{\varphi}(z)$. We make a change of the function $\widetilde{\varphi}(z)$, putting

$$\widetilde{\varphi} = \chi(z)\varphi_1(z),$$

where $\chi(z)$ is the analytic function constructed in 3°. Then $\varphi_1(z)$ is holomorphic in D, and moreover $\varphi_1(0) = 0$. Introduce the function $\varphi_1(z)$ into condition (4). We obtain

$$\mathrm{Re}\left\{\partial_t \varphi_1(t) + \left[\partial_t \chi + \partial_t \ln \sqrt{aK}\sqrt{\overline{K}}\,\chi\right] \varphi_1(t) + \tan\alpha \cdot c^2(s) \varphi_1(t)\right\} = 0, \quad (6)$$

where $c^2(s) = \lambda(t)\chi(t)$ is a real function.

Since $c^2(s) \neq 0$, for sufficiently large $\tan\alpha$ one may assume that

$$\operatorname{Re}\left\{\partial_t \chi + \partial_t \ln \sqrt{aK}\sqrt{\overline{K}}\right\} + \tan\alpha\, c^2(s) \equiv c_1^2(s,\alpha) \neq 0.$$

Next denote

$$-c_2(s) = \operatorname{Im}\left\{\partial_t \chi + \partial_t \ln \sqrt{aK}\sqrt{\overline{K}}\right\},$$

and rewrite condition 6 in the form

$$a^{(l)} \frac{\partial U}{\partial l_t} + c_1^2(s,\alpha) U = c_2(s) V,$$

where $a^{(l)} \neq 0$, $a^{(l)} > 0$; $\partial/\partial l_t$ is the derivative in the direction $\{l^1, l^2\}$; $U + iV = \varphi_1(z)$, $U(0,0) = V(0,0) = 0$.

Consider the relation

$$\iint_D U \Delta U \, du^1\, du^2 = \int_\Gamma U U_{u^1}\, du^1 - U U_{u^2}\, du^2 - \iint_D (\nabla U)^2\, du^1\, du^2.$$

For harmonic functions we have

$$\iint_D (\nabla U)^2\, du^1\, du^2 = \int_\Gamma U \frac{\partial U}{\partial n}\, ds.$$

Since $\{l^1, l^2\} \nparallel \{t^1, t^2\}$, we have

$$\frac{\partial U}{\partial l_t} = \cos\psi \frac{\partial U}{\partial n} + \sin\psi \frac{\partial U}{\partial s},$$

where $\cos\psi \neq 0$, $\psi \in C^{1,\alpha}$, $0 < \alpha < 1$. Therefore we have

$$\iint_D (\nabla U)^2\, du^1\, du^2 = \int_\Gamma \frac{U}{\cos\psi} \left(\frac{\partial U}{\partial l_t} - \sin\psi \frac{\partial U}{\partial s} \right) ds =$$

$$= \int_\Gamma \frac{U}{\cos\psi} \left[-\frac{c_1^2(s,\alpha)}{a^{(l)}} U - \frac{c_2(s)}{a^{(l)}} V \right] ds + \int_\Gamma (-\tan\psi)\, U \frac{\partial U}{\partial s}\, ds.$$

Transforming the contour integrals in this expression, we obtain

$$\iint\limits_D (\nabla U)^2\, du^1\, du^2 = -\int\limits_\Gamma A_1^2 U^2\, ds - \int\limits_\Gamma \operatorname{tg}\psi\, U \frac{\partial U}{\partial s}\, ds + \int\limits_\Gamma U V B_1\, ds, \qquad (7)$$

where $A_1^2 = c_1^2(s,\alpha)/a^{(l)}$, $B_1 = -c_2(s)/a^{(l)}$. Further, we have

$$\int\limits_{\Gamma} \operatorname{tg}\psi \cdot U\,\frac{\partial U}{\partial s}\,ds = \frac{1}{2}\int\limits_{\Gamma} \operatorname{tg}\psi \cdot \frac{\partial U^2}{\partial s}\,ds = -\frac{1}{2}\int\limits_{\Gamma} \frac{\partial \operatorname{tg}\psi}{\partial s}\,U^2\,ds.$$

Therefore

$$\left|\int\limits_{\Gamma} \operatorname{tg}\psi \cdot \frac{\partial U}{\partial s}\,U\,ds\right| \le \mu_1 \int\limits_{\Gamma} U^2\,ds, \qquad \text{where } \mu_1 = \text{const} < \infty. \tag{8}$$

From (7), by virtue of (8), we obtain the estimate

$$\left|\int\limits_{\Gamma} A_1^2 U^2\,ds\right| \le \mu_1 \int\limits_{\Gamma} U^2\,ds + \left|\int\limits_{\Gamma} UVB_1\,ds\right| \le \mu_1 \|U\|_{L_2} + \mu_2 \left|\int\limits_{\Gamma} |U|\cdot|V|\,ds\right|, \tag{9}$$

where $\mu_2 = \max|B_1|$.

Consider the last term in inequality (9). Since the functions U, V are harmonically conjugate and $U(0,0) = V(0,0) = 0$, we have $\|U\|_{L_2} = \|V\|_{L_2}$, and therefore

$$\int\limits_{\Gamma} |U|\cdot|V|\,ds \le \|U\|_{L_2}^2.$$

Hence (9) may be rewritten as

$$\int\limits_{\Gamma} A_1^2 U^2\,ds \le (\mu_1 + \mu_2)\|U\|_{L_2}^2. \tag{10}$$

Let us estimate the integral on the left in inequality (10). Since $A_1^2 \neq 0$ on Γ, denoting $\mu_3 = \min A_1^2$, we obtain

$$\mu_3 \|U\|_{L_2}^2 \le (\mu_1 + \mu_2)\|U\|_{L_2}^2.$$

By assumption $\|U\|_{L_2} \neq 0$, and therefore from this we obtain

$$\mu_3 \le \mu_1 + \mu_2. \tag{11}$$

But $\mu_3 = \min A_1^2(s,\alpha)$ is a function of the angle α, $\alpha \in (\pi/2 - \varepsilon, \pi/2)$, and $\mu_3 \to \infty$ as $\varepsilon \to 0$; on the other hand, μ_3 admits an upper bound independent of α. The contradiction obtained proves the theorem.

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Note: Figure translations are in progress. See original paper for figures.

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