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Abstract

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O. I. ZAV' YALOV, V. N. SUSHKO

ON A CLASS OF REPRESENTATIONS OF THE CANONICAL COMMUTATION RELATIONS OF QUANTUM FIELD THEORY

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The problems of relativistic quantum field theory and of the quantum theory of many bodies lead to the necessity of considering various mutually nonequivalent representations of the canonical commutation relations (c.c.r.). In this connection, representations of a special kind are of considerable interest—representations in the tensor product of a countable family of Hilbert spaces ([1]), in each of which the Schrödinger representation of the c.c.r. for one degree of freedom is realized ([2,3]). Such representations we shall call direct product representations (d.p.r.).

It is known ([4]) that an arbitrary representation of the c.c.r. can be realized in the direct integral $\int_\Gamma \oplus \mathcal{H}(a)\, d\mu(a)$ of Hilbert spaces $\mathcal{H}(a)$ over the set $\Gamma = \{a\}$ of all sequences $a = [a_1, a_2, ...]$ of nonnegative integers a_i and is completely determined by a triple $\{\nu(a), \mu(a), [c_k(a)]\}$. Here $\nu(a)$ is the dimension of the space $\mathcal{H}(a)$, μ is a quasi-invariant measure on Γ, and $[c_k(a)]$ is a family of unitary operators on $\mathcal{H}(a)$ satisfying certain simple restrictions ([4]).

In the present note we formulate a number of results on the structure of d.p.r.' s and on their place in the scheme of the general classification of Gårding and Wightman ([4]).

Let the Hilbert space $\mathcal{H} = \prod_{k=1}^{\infty} \otimes \mathcal{H}_k$ be the tensor product of the family $\{\mathcal{H}_k = L^2(x)\}_{k=1}^{\infty}$ of a countable number of copies of the space $L^2(x)$. Consider the separable complex Hilbert space $\mathfrak{F}$ of basic functions and the basis $\{e_i\}_{i=1}^{\infty}$ in it. Let $\mathfrak{F}$ be the set of finite linear combinations of elements of the basis. A direct product representation of the c.c.r. is a Weyl mapping $W : \mathfrak{F} \to \mathcal{U}(\prod \otimes \mathcal{H}_k)$ of the manifold $\mathfrak{F}$ into the group of unitary operators on $\prod \otimes \mathcal{H}_k$ such that

$$W(f) = W\left(\sum z_k e_k\right) = \prod_k \otimes W_k(z_k), \tag{1}$$

where $W_k(z)$ is a unitary operator in $L^2(x)$, defined by the equality

$$(W_k(z)\varphi)(x) = \exp i[{}^1/_2\xi\eta + \xi x] \cdot \varphi(x+\eta). \tag{2}$$

It follows from (1) that a d.p.r. is orthogonally reduced by each incomplete tensor product $\mathcal{H}^\chi = \prod_k^\chi \otimes \mathcal{H}_k$, generated by a product vector $\chi = \prod_k \otimes \chi_k$ with $\chi_k \in \mathcal{H}_k = L^2(x)$ and $\|\chi_k\| = 1$. It can be shown that the restriction W^χ of the representation W to the subspace $\mathcal{H}^\chi$ is irreducible and that the bicommutant $\{W(f)\}''_{f\in\mathfrak{F}}$ of the set $\{W(f)\}_{f\in\mathfrak{F}}$ coincides with the closure $\mathfrak{B}$ (in the weak operator topology) of the algebra generated by

operators of the form $A = I_1 \otimes I_2 \ldots \otimes A_k \otimes I_{k+1} \ldots$, where A_k is any bounded operator on $\mathcal{H}_k$, and I is the identity operator.

Theorem 1. *Two representations W^χ and $W^{\chi'}$ are equivalent if and only if*

$$\sum_k |(\chi_k, \chi_k') - 1| < \infty, \tag{3}$$

i.e., when $\mathcal{H}^\chi$ is weakly equivalent to $\mathcal{H}^{\chi'}$ in the sense of von Neumann (1).

Proof. For brevity we consider only the necessity of condition (3). Let W^χ be equivalent to $W^{\chi'}$, i.e. for all $f \in \mathfrak{F}$

$$W^\chi(f) = UW^{\chi'}(f)U^{-1}. \tag{4}$$

Suppose that condition (3) is not satisfied. Then, according to von Neumann' s results (Theorem X of (1)), there exists an operator

$$A \in \{W(f)\}''_{f\in\mathfrak{F}} \tag{5}$$

such that the restrictions of A to $\mathcal{H}^\chi$ and $\mathcal{H}^{\chi'}$, respectively, are

$$A^\chi = I, \qquad A^{\chi'} = 0. \tag{6}$$

But from (4) and (5) it follows that $A^\chi = UA^{\chi'}U^{-1}$, which contradicts (6). By another, more cumbersome method this result was obtained earlier in (3).

We now clarify the structure of triples $\{\nu, \mu, [c_k]\}$ corresponding to irreducible p.p.p. Note that every such representation W^χ generates a commuting family $\{N_j^\chi\}_{j=1}^\infty$ of self-adjoint particle-number operators

$$N_j^\chi = (I_1 \otimes \ldots \otimes n_j \otimes \ldots)^\chi,$$

where

$$(n_j\varphi)(x) = \tfrac{1}{2}[-d^2\varphi(x)/dx^2 + x^2\varphi(x) - \varphi]. \tag{7}$$

From (7) it is seen that the spectrum of N_j^χ is the set $S_j = \{0, 1, 2, ...\}$, so that the spectral decomposition of N_j^χ has the form

$$N_j^\chi = \sum_{a_j \in S_j} a_{jP} j(a_j).$$

Using the family $\{P_j(a_j)\}_{j=1}^{\infty}$, one can construct a spectral projection-valued measure M on the σ-ring Σ of subsets $E \subset \Gamma$, where $\Gamma = \{a\}$ is the Cartesian product $\prod S_j$ of the spectra S_j. Here the operator-valued measure M is a product measure, i.e. if $E \in \Sigma$ and $E = \prod E_j$, where $E_j \subset S_j$ for all j, then

$$M(E) = \prod_j M_j(E_j), \tag{8}$$

where

$$M_j(E_j) = \sum_{a_j \in E_j} P_j(a_j), \tag{9}$$

and the product (8) converges in the strong operator topology.

Next note that in each $\mathscr{H}^\chi$ one can choose a product vector

$$\chi = \prod_k \otimes \chi_k$$

such that:

a) the incomplete tensor product $\mathscr{H}^\chi$ generated by χ coincides with $\mathscr{H}^{\tilde{\chi}}$;

b) χ is a cyclic vector with respect to the family $\{N_j^\chi\}$, and all coefficients $q_k^m = (\chi_k, h_m)$ in the expansion of χ_k in Hermite functions $h_m(x)$ are nonzero.

Using the measure M and the cyclic product vector $\chi = \prod_{\mathscr{H}^\chi} \otimes \chi_k$, one can in the usual way isometrically map $\mathscr{H}^\chi$ into

$$\int_T \oplus \mathscr{H}(\alpha)\, d\mu(\alpha),$$

where $\dim \mathscr{H}(\alpha) = 1$. In this case the operators N_j^χ pass into the operators of multiplication by α_j, and the representation W^χ assumes the standard form (4). The numerical measure $\mu(E) = (M(E)\chi, \chi)$, by virtue of the choice of the cyclic vector and property (8), turns out to be a quasi-invariant product measure generated by the set of numbers

$$p_j^m = (P_j(m)\chi, \chi) = |q_k^m|^2 \tag{10}$$

with q_k^m from b). It is not difficult to find also the form of the last element of the triple—the operators $c_k(\alpha)$. We have

$$c_k(\alpha) = \exp -i[\theta_k(\alpha_k) - \theta_k(\alpha_k - 1)], \tag{11}$$

where $\theta_k(m) = \arg q_k^m$.

Conversely, it is not difficult to verify that for any triple $\{v(\alpha) = 1, \mu,\ [c_k(\alpha) = e^{i\Phi_k(\alpha_k)}]\}$ with a product measure μ there is a corresponding irreducible D.P.R. Thus, we have proved

Theorem 2. *A representation of the c.c.r. is equivalent to some irreducible direct-product representation if and only if $v = 1$, μ is an (equivalent) product measure, and $c_k(\alpha)$ have the form (11).*

Combining Theorems 1 and 2 and using formula (10), we obtain

Corollary. *Two quasi-invariant product measures μ and μ', generated respectively by sets of numbers p_k^m and $p_k'^m$, are equivalent if and only if*

$$\sum_k \left| 1 - \sum_m \sqrt{p_k^m p_k'^m} \right| < \infty. \tag{12}$$

This result is an exhaustive solution of the corresponding problem posed in (4), and was first obtained in (5).

The class of representations considered makes it possible to give a simple criterion for the propriety of canonical finite-particle transformations (not necessarily linear). We have in mind the following. Let a_k^+ and a_k be the creation and annihilation operators corresponding to some representation of the c.c.r. in the space $\mathcal{H}$. A transformation of the family $\{a_k, a_k^+\}_{k=1}^{\infty}$ into itself will be called a finite-particle canonical transformation (c.t.) if for all k the pair $\langle a_k', a_k'^+ \rangle$ is represented in the form

$$\langle a_k', a_k'^+ \rangle = \langle f_k(a_{k_1} \dots a_{k_n}; a_{k_1}^+ \dots a_{k_n}^+), \bar{f}_k(a_{k_1}^+ \dots a_{k_n}^+, a_{k_1} \dots a_{k_n}) \rangle, \tag{13}$$

where $n < \infty$, and

$$[a_k', a_k'^+] \equiv a_k' a_k'^+ - a_k'^+ a_k' = [a_k, a_k^+],$$

so that $\{a_k', a_k'^+\}_{k=1}^{\infty}$ generates in $\mathcal{H}$ a new representation of the c.c.r., generally speaking not equivalent to the original one. If these two representations are equivalent, then the c.t. is called proper. Let us note at once the not always clearly understood fact that propriety of a c.t. depends not only on the form of the functions f_k in (14), which reflect only the algebraic properties of the c.t., but also on the properties of the original representation in $\mathcal{H}$. In the case when the latter is equivalent to an irreducible D.P.R. W^χ, the transformed representation is equivalent to some other D.P.R., namely $W^{U\chi}$, so that, by Theorem 1, the criterion for propriety of the c.t. (for brevity we restrict ourselves to one-particle c.t.' s) has the form

$$\sum_k |(U_k\chi_k, \chi_k) - 1| < \infty. \tag{14}$$

Here $U\chi = \prod_k \otimes U_k\chi_k$, and each U_k is a unitary operator in $\mathcal{L}^2$ such that for all k

$$\langle a'_k, a'^+_k \rangle = \langle \widetilde{U}_k a_k \widetilde{U}_k^{-1}, \widetilde{U}_k a_k^+ \widetilde{U}_k^{-1} \rangle,$$

where $\widetilde{U}_k = I_1 \otimes \dots \otimes U_k \otimes I_{k+1} \dots$.

An analogous criterion is easily derived also for finite-frequency canonical transformations. It is also of interest to note that for any such canonical transformation there exists a representation in which it is proper.

All the results obtained above, with obvious modifications, carry over also to the case of canonical anticommutation relations.

Steklov Mathematical Institute
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.

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