A ONE-PRODUCT DYNAMIC MODEL WITH A CONSTRAINED STRUCTURE

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Abstract

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A ONE-PRODUCT DYNAMIC MODEL WITH A CONSTRAINED STRUCTURE

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1. We consider the following scheme ([1]): fixed assets have structure λ, if per unit of labor resources there are λ units of fixed assets, so that $\lambda = K/T$; $K(t)$ is the fixed assets at time t, and $T(t)$ the labor resources at time t. Let $\varphi(\lambda)$ be the spectrum of the distribution of labor resources over assets, so that $\varphi(\lambda)d\lambda$ determines the number of units of labor power associated with assets of structure from λ to $\lambda + d\lambda$. The service life of the assets is taken to be infinite; in this case they do not wear out and are not transformed. The newly created product, after subtraction of the share going to consumption, is used to create new assets, which makes it possible to increase the organic structure of capital; labor for new assets, in addition to the increment of its resources, is allocated by removing a certain number of workers from assets of the lowest organic structure. The assets thus freed remain, as a rule, unused thereafter.

Let $m(t)$, $M(t)$ be the bounds of the spectrum of the structure of assets at time t. We present the derivation, due to L. V. Kantorovich, of the equations describing this process. The free resources by the time $t + dt$ will be

$$\Delta T = T'(t)dt + \varphi(m(t))dm(t); \tag{1}$$

since the labor requirement is equal to $\varphi(M(t))dM(t)$, comparing with (1), we write the labor-balance equation

$$\varphi(M(t))dM(t) = T'(t)dt + \varphi(m(t))dm(t). \tag{2}$$

The net output produced at time t will be

$$\int_m^M U(\lambda\varphi(\lambda), \varphi(\lambda))d\lambda = U(t),$$

where $U(\lambda\varphi(\lambda), \varphi(\lambda))d\lambda$ is the quantity of net output produced on assets of structure λ at time t.

The production function U is assumed to be concave in each variable and positively homogeneous of degree one. Further, since the fund of free capital investments by the time $t+dt$ is equal to $d\Phi = [U(t)-V(t)]dt$, $V(t)$ being consumption, and the requirement for capital investments is equal to $M\varphi(M)dM$, the balance equation for assets takes the form

$$d\Phi = M(t)\varphi(M)dM. \tag{3}$$

From equations (2) and (3) we directly obtain the increment of net output over the time dt

$$\begin{aligned} dU(t) &= U(M(t),1)\varphi(M(t))dM(t) - U(m(t),1)\varphi(m(t))dm(t) \\ &= [U(M(t),1) - U(m(t),1)]d\Phi/M(t) + U(m(t),1)T'(t)dt. \end{aligned}$$

Next, the differential step along the spectrum must be such that the increment of output is greatest (differential optimization) and,

therefore, the function

$$\frac{1}{M(t)}[U(M(t),1) - U(m(t),1)] \tag{4}$$

must be maximal.

Differentiating (4) with respect to M and equating the resulting expression to zero, we obtain the condition for the bounds of the spectrum

$$\frac{\partial U(M,1)}{\partial M}M - U(M,1) + U(m,1) = 0. \tag{5}$$

Thus, the system of equations for the functions $\varphi(\lambda)$, $M(t)$ finally takes the form

$$\int_m^M \varphi(\lambda)\, d\lambda = T(t); \tag{6}$$

$$M\varphi(M)M' = \int_m^M U(\lambda\varphi(\lambda), \varphi(\lambda))\, d\lambda - V(t). \tag{7}$$

We shall make the following assumptions:

1) the function $U(x,y) = x^{\alpha}y^{1-\alpha}$, $0 < \alpha < 1$ (the Cobb–Douglas function);

2) $V(t) = (1-\gamma)U(t)$, $0 < \gamma < 1$;

3) $T(t) = T(0)e^{\delta t}$, δ is the demographic growth rate.

Under these assumptions, equations (5)–(7) can be written in the form

$$m(t) = \beta M(t), \qquad \beta = (1-\alpha)^{1/\alpha};$$

$$\int_{\beta M}^{M} \varphi(\lambda)\, d\lambda = T(t); \tag{8}$$

$$M\varphi(M)M' = \gamma \int_{\beta M}^{M} \lambda^{\alpha}\varphi(\lambda)\, d\lambda. \tag{9}$$

We shall say that the functions $M, \varphi(\lambda)$ are functions of the system (8)–(9) if they satisfy equations (8), (9). Let us study the asymptotic behavior of these functions as $t \to \infty$.

Theorem 1. *If M is a function of the system (8)–(9), then M is bounded and, moreover, there exists a finite limit*

$$\lim_{t\to\infty} M(t) = c < \infty.$$

Proof. From equation (9) we have $M'(t) > 0$ and, consequently, the function $M(t)$ increases. Thus, $\lim_{t\to\infty} M(t)$ exists, and it remains to prove that it is finite. Using equations (8)–(9) and the monotonicity of the function $M(t)$, it is easy to obtain the inequalities

$$M\varphi(M)M' \le \gamma M^{\alpha} T(t), \qquad \varphi(H) \ge T'(t)/M'(t),$$

whence it follows directly that

$$M(t) \le (\gamma/\delta)^{1/(1-\alpha)}. \tag{10}$$

The theorem is proved.

Theorem 2. *If M and $\varphi(\lambda)$ are functions of the system (8)–(9), then the limit $\lim_{t\to\infty} \varphi(\beta M)/\varphi(M)$ exists and is equal to*

$$\lim_{t\to\infty} \frac{\varphi(\beta M)}{\varphi(M)} = \frac{1}{\beta} \cdot \frac{\gamma c^{\alpha-1} - \delta}{\gamma(1-\alpha)c^{\alpha-1}}. \tag{11}$$

Proof. Differentiating equations (8)–(9), we obtain

$$\varphi(M)M' - \beta\varphi(\beta M)M' = T'; \tag{12}$$

$$\left[M\varphi(M)M'\right]_t' = \gamma\{M^\alpha\varphi(M)M' - \beta^{1+\alpha}M^\alpha\varphi(\beta M)M'\}. \tag{13}$$

Equation (12) can be rewritten in the equivalent form

$$M\varphi(M)M' = \frac{T'M}{1 - \beta\varphi(\beta M)/\varphi(M)}. \tag{14}$$

Putting

$$\chi(t) = \frac{M(t)}{1 - \beta\varphi(\beta M)/\varphi(M)}$$

and taking (14) into account, we write (13) in the equivalent form

$$[T'\chi(t)]' = \gamma\{M^{\alpha-1}T'\chi(t) - \beta^\alpha M^{\alpha-1}T'\chi(t) + \beta^\alpha M^\alpha T'\},$$

whence it follows that

$$\chi'(t)T' + \delta\chi(t)T' = \gamma(1-\beta^\alpha)M^{\alpha-1}T'\chi(t) + \gamma\beta^\alpha M^\alpha T'$$

or, equivalently,

$$\chi'(t) + [\delta - \gamma(1-\beta^\alpha)M^{\alpha-1}]\chi(t) = \gamma\beta^\alpha M^\alpha. \tag{15}$$

Further, since $\delta - \gamma(1-\beta^\alpha)M^{\alpha-1} > 0$ (by virtue of inequality (10)) and $\lim_{t\to\infty} M(t) = c$, every solution of equation (15) (see ([2])) tends to

$$\frac{\gamma\beta^\alpha c^\alpha}{\delta - \gamma(1-\beta^\alpha)c^{\alpha-1}},$$

and, consequently,

$$\lim_{t\to\infty}\frac{\varphi(\beta M)}{\varphi(M)} = \frac{1}{\beta} - \frac{\gamma c^{\alpha-1} - \delta}{\gamma(1-\alpha)c^{\alpha-1}}.$$

The theorem is proved.

Theorem 3. *If M is a function of system (8)–(9), then*

$$c = (\gamma/\delta)^{1/(1-\alpha)}.$$

Proof. From inequality (10) we have

$$c \leqslant (\gamma/\delta)^{1/(1-\alpha)}, \tag{16}$$

and from equation (14), taking (11) into account, we obtain

$$\lim_{t\to\infty} \frac{T'}{M\varphi(M)} = \frac{\delta - \alpha\gamma c^{\alpha-1}}{\gamma(1-\alpha)c^{\alpha-1}}. \tag{17}$$

Next, transforming the inequality

$$M\varphi(M)M' \leqslant \gamma M^{\alpha} T$$

and passing to the limit with (17) taken into account, we obtain

$$c^{1-\alpha}\frac{\delta}{\gamma} \leqslant \frac{\delta - \alpha\gamma c^{\alpha-1}}{\gamma(1-\alpha)c^{\alpha-1}},$$

whence we get

$$c \geqslant (\gamma/\delta)^{1/(1-\alpha)}. \tag{18}$$

Comparing (16) and (18), we obtain $c = (\gamma/\delta)^{1/(1-\alpha)}$. The theorem is proved. From Theorems 2 and 3 we obtain the corollary

$$\lim_{t\to\infty} \frac{\varphi(\beta M)}{\varphi(M)} = 0.$$

Theorem 4. If $M, \varphi(M)$ are functions of the system (8)–(9), then the asymptotic formula holds

$$\frac{1}{\delta} = \lim_{t\to\infty} \frac{1}{c - M(t)} \int_t^{\infty} \frac{T(\tau)d\tau}{\varphi(M(\tau))}, \qquad c = (\gamma/\delta)^{1/(1-\alpha)}.$$

Proof. From equation (12) we have

$$M' = \frac{T'(t)}{\varphi(M)[1 - \beta\varphi(\beta M)/\varphi(M)]}. \tag{19}$$

Next, integrating (19) from t to $+\infty$,

$$c - M(t) = \delta \int_t^{\infty} \frac{T(\tau)d\tau}{\varphi(M(\tau))[1 - \beta\varphi(\beta M)/\varphi(M)]}, \tag{20}$$

and passing to the limit in (20), we obtain

$$\lim_{t\to\infty}\int_t^\infty \frac{T(\tau)d\tau}{\varphi(M(\tau))\left[1-\beta\frac{\varphi(\beta M)}{\varphi(M)}\right]}=0.$$

Using l' Hôpital' s rule, taking into account the corollary to Theorems 2 and 3, we shall have

$$\lim_{t\to\infty}\int_t^\infty \frac{T(\tau)d\tau}{\varphi(M)[1-\beta\varphi(\beta M)/\varphi(M)]}\Big/\int_t^\infty \frac{T(\tau)d\tau}{\varphi(M(\tau))}=1. \tag{21}$$

Comparing (20) and (21), we obtain the theorem.
It follows from Theorem 4 that:
If $\varphi(M)$ is a function of the system (8)–(9), then

$$\int^\infty \frac{T(t)}{\varphi(M(t))}\,dt<\infty.$$

2. The magnitude of the norm of efficiency of capital investments under the conditions of the given model characterizes the increase in production of output per unit time corresponding to a unit of additional capital investment, and is equal to $n_{\mathrm{e}}=U'(M(t),t)$. In the Cobb–Douglas case the formula for the norm of efficiency takes the form

$$n_{\mathrm{e}}=aM^{\alpha-1}. \tag{22}$$

Passing to the limit in (22), we obtain

$$\lim_{t\to\infty} n_{\mathrm{e}}=\alpha\delta/\gamma. \tag{23}$$

It follows from formula (23) that the asymptotic behavior of the norm of efficiency is the same as in the case of the model previously studied by us in paper (3).

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CITED LITERATURE

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[2] E. Kamke, *Handbook of Ordinary Differential Equations*, "Nauka," 1965.
[3] L. V. Kantorovich, I. G. Globenco, *DAN*, **174**, No. 3 (1967).

Note: Figure translations are in progress. See original paper for figures.