ON THE THEORY OF ORLICZ SPACES

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 513.88 : 513.881

MATHEMATICS

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ON THE THEORY OF ORLICZ SPACES

(Presented by Academician A. N. Tikhonov on 21 XII 1966)

1°. Introduction. In this note several new results from the theory of Orlicz spaces in the sense of Zaanen are formulated. We give the necessary information about these spaces ([1,2]).

A function $M(u)$, $0 \le u \le \infty$, $0 \le M(u) \le \infty$, is called a **Young function** if it satisfies the following conditions: $M(u)$ is nondecreasing on $[0, \infty]$, with $M(0) = 0$, $M(\infty) = \infty$; $0 < d_M \le \infty$, where $d_M = \sup\{u \mid M(u) < \infty\}$; $0 \le c_M < \infty$, where $c_M = \sup\{u \mid M(u) = 0\}$; $M(u)$ is continuous for every $u < d_M$; $M(u)$ is continuous from the left at $u = d_M$; $M(u)$ is convex on the interval $[0, d_M)$.

Definition. $M(u)$ satisfies the Δ_β-condition (where $1 < \beta < \infty$) on $[a, b]$, $0 \le a < b \le \infty$, if there exists a K, $0 \le K < \infty$, such that $M(\beta u) \le KM(u)$ for $a \le u \le b$.

Lemma. If $d_M < \infty$, then the following two conditions are equivalent:
1) $M(d_M) < \infty$; 2) there exist $\beta > 1$ and $u_0 \in [0, \beta^{-1} d_M)$ such that $M(u)$ satisfies the Δ_β-condition on $[u_0, \beta^{-1} d_M]$.

Let X be a nonempty set; R, some σ-algebra of its subsets; μ, a σ-finite measure ([3]) defined on R, with $0 < \mu X \le \infty$. Fix an increasing sequence of sets $X_n \in R$ such that $0 < \mu X_n < \infty$, $n = 1, 2, ...$, and

$$X = \bigcup_{n=1}^{\infty} X_n.$$

Following ([2]), we shall call a set $E \subset X$ **bounded** if $E \subset X_n$ starting from some n. We shall assume that the measure μ is complete (i.e., if $\mu A = 0$ and $B \subset A$, then $B \in R$) and continuous (i.e., if $\mu A > 0$ and $0 < \delta < \mu A$, then there exists a $B \in R$ such that $B \subset A$ and $\mu B = \delta$).

Let S be the set of all functions measurable on X with respect to the measure μ, taking values in the extended real line or in the extended complex plane. Functions which coincide almost everywhere on X are regarded as identical.

The **Young class** is the set

$$P_M = \left\{ u \in S \mid \int_X M[|u(x)|]\, dx < \infty \right\}.$$

Obviously, P_M is contained in the set of almost everywhere finite functions from S and is a convex set.

The **Orlicz space** L_M is the union of all sets aP_M, $0 < a < \infty$. L_M is a vector space (under certain conditions the Young class P_M itself is a vector space; then $L_M = P_M = aP_M$, $0 < a < \infty$), and after the introduction of the norm

$$\|u\|_1 = \inf \left\{ \alpha > 0 \mid \int_X M[\alpha^{-1}|u(x)|]\, dx \leq 1 \right\}$$

or

$$\|u\|_2 = \sup \left\{ \int_X |u(x)v(x)|\, dx \mid \int_X N[|v(x)|]\, dx \leq 1 \right\}$$

(where $N(v) = \max\{uv - M(u) \mid 0 \leq u < \infty\}$, $0 \leq v \leq \infty$) it becomes a Banach space.

The norms $\| \ \|_1$ and $\| \ \|_2$ are equivalent; more precisely, $\|u\|_1 \leq \|u\|_2 \leq 2\|u\|_1$. Both norms are defined on the whole set S, but if $u \in S \setminus L_M$, then $\|u\|_k = \infty$, $k = 1, 2$. We note that

$$\{u \mid \|u\|_1 \leq 1\} = \left\{ u \,\middle|\, \int_X M[|u(x)|]\, dx \leq 1 \right\} \subset P_M.$$

An important role in the theory of the space L_M is played by its closed subspace L_M^f, defined as the intersection of all the sets aP_M, $0 < a < \infty$. Here $L_M^f = \{\theta\}$ (where θ is the zero function) if and only if $d_M < \infty$. On the other hand, $L_M^f = L_M$ if and only if $L_M = P_M$.

2°. Topological and metric properties of the Young class as a subset of an Orlicz space. In the following Theorems 1-3, certain results obtained in ([4]) for a narrower class of Orlicz spaces are generalized and supplemented (in particular, in ([4]) it is assumed that $c_M = 0$, $d_M = \infty$, and $\mu X < \infty$).

Theorem 1. *The following three conditions are pairwise equivalent:*

A. *The Young class P_M is a closed set in the Orlicz space L_M.*

B. $P_M = \{u \in L_M \mid |u(x)| \leq d_M$ *almost everywhere on* $X\}$.

C. *The function $M(u)$ satisfies the Δ_β-condition on $[u_0, \beta^{-1} d_M]$ for some $\beta > 1$ and $u_0 \in [0, \beta^{-1} d_M)$, and, if $\mu X = \infty$, then $u_0 = 0$.*

Corollary 1. *If $d_M = \infty$, then P_M is closed if and only if $L_M = P_M$.*

Remark. From Theorem 1, in particular, there follows the well-known criterion for the coincidence of L_M with P_M: $L_M = P_M$ if and only if $d_M = \infty$ and $M(u)$ satisfies the Δ_2-condition on $[u_0, \infty]$, where $u_0 = 0$ if $\mu X = \infty$.

Theorem 2. *The Young class P_M is an open set in the space L_M if and only if $L_M = P_M$.*

For $u \in L_M$, put

$$\rho_k(u, L_M^f) = \inf\{\|u - v\|_k \mid v \in L_M^f\}, \qquad \Pi_k = \{u \in L_M \mid \rho_k(u, L_M^f) < 1\},$$

$k = 1, 2$.

Theorem 3. *Let $d_M = \infty$. Then $\Pi_k = \operatorname{int} P_M$, $\overline{\Pi}_k = \overline{P}_M = \{u \in L_M \mid \rho_k(u, L_M^f) \leq 1\}$, $k = 1, 2$.*

Corollary 2. *If $d_M = \infty$, then*

$$\rho_1(u, L_M^f) = \rho_2(u, L_M^f) = \inf\{a > 0 \mid u \in aP_M\}$$

for every $u \in L_M$.

Corollary 2 means that if $d_M = \infty$, then the norms $\| \, \|_1$ and $\| \, \|_2$ induce one and the same norm in the factor space L_M / L_M^f.

Corollary 3. *If $d_M = \infty$ and $L_M \neq P_M$, then the maximal radius of balls contained in P_M is equal to one both for the norm $\| \, \|_1$ and for the norm $\| \, \|_2$.*

Theorem 4.

$$\sup\{\|u\|_k \mid u \in P_M\} = \|d_M\|_k, \qquad k = 1, 2.$$

Here d_M denotes the function taking the value d_M at all $x \in X$.

Corollary 4. *P_M is a bounded set in the space L_M if and only if $d_M \in L_M$.*

From Corollary 4 there follows the following **boundedness criterion for** P_M: if $d_M = \infty$, then P_M is unbounded; if $d_M < \infty$, then P_M is bounded if and only if $\mu X < \infty$, or $\mu X = \infty$ and $c_M > 0$.

Theorem 5. *Let $d_M < \infty$. Then*

$$\max\{r \mid D_r \subset P_M\} = 1,$$

where $D_r = \{u \mid \|u\|_1 \leq r\}$. Moreover, if $M(d_M)\mu X \leq 1$, then $\|d_M\|_1 = 1$ and $P_M = D_1$; if $M(d_M)\mu X > 1$, then $\|d_M\|_1 > 1$ and both inclusions

$$D_1 \subset P_M \subset \{u \mid \|u\|_1 \leq \|d_M\|_1\}$$

are strict.

3°. On some subspaces of an Orlicz space. Following ([2]), denote by L_M^b the closure in the space L_M of the set of bounded functions $u(x)$ for which the set $\{x \in X \mid u(x) \neq 0\}$ is bounded. In addition, denote by E_M^f the closure in L_M

sets of bounded functions $u(x)$ for which $\mu\{x \in X \mid u(x) \neq 0\} < \infty$, and by E_M ([4]) the closure in L_M of the set of bounded functions contained in L_M (one should take into account that if $\mu X = \infty$ and $c_M = 0$, then, for example, identically constant functions different from zero do not belong to L_M).

As shown in ([2]), $L_M^f \subset L_M^b$, and $L_M^f = L_M^b$ if and only if $d_M = \infty$. On the other hand, it is obvious that $L_M^b \subset E_M^f \subset E_M \subset L_M$.

Theorem 6. If $d_M = \infty$, then $L_M^b = E_M^f$ ([2]). If $d_M < \infty$, then $L_M^b = E_M^f$ if and only if $\mu(X \setminus X_n) = 0$, starting from some n.

Theorem 7. If $\mu X < \infty$, then $E_M^f = E_M$. If $\mu X = \infty$, then $E_M^f = E_M$ if and only if $c_M = 0$ and $M(u)$ satisfies the Δ_2-condition on $[0, b]$ for some $b > 0$.

Theorem 8. If $d_M < \infty$, then $E_M = L_M$. If $d_M = \infty$, then $E_M = L_M$ if and only if $M(u)$ satisfies the Δ_2-condition on $[u_0, \infty]$ for some $u_0 > 0$.

4°. Convergence in mean. It is said ([4]) that a sequence $u_n \in L_M$ **converges in mean** to $u \in L_M$ if

$$\lim_{n\to\infty} \int_X M[|u_n(x) - u(x)|]\, dx = 0.$$

From the definition of the norm $\| \ \|_1$ it follows easily that convergence in norm entails convergence in mean.

Theorem 9. Convergence in norm in the space L_M is equivalent to convergence in mean if and only if $L_M = P_M$ and $c_M = 0$.

It should be noted that Theorems 8 and 9 generalize the corresponding results from ([4]).

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Received
12 XII 1966

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