ON THE SPECTRUM OF NON-SELF-ADJOINT DIFFERENTIAL OPERATORS

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Abstract

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MATHEMATICS

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ON THE SPECTRUM OF NON-SELF-ADJOINT DIFFERENTIAL OPERATORS

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1. In the papers ([1],[2]), the eigenvalues of the operator l_h (in $L_2 = L_2(0,\infty)$) were studied:

$$ly = -y'' + q(x)y, \qquad y'(0) - hy(0) = 0,$$

where h is a complex number, $q(x)$ a complex-valued function. The spectral analysis of the operator L_h was carried out by M. A. Naimark ([1]). In particular, he proved that if $q(x)$ satisfies the condition

$$\sup_x |q(x)| \exp(\varepsilon x) < \infty, \qquad \varepsilon > 0, \tag{1}$$

then the operator l_h has a finite number of eigenvalues. B. S. Pavlov ([2]) obtained a more precise result, namely: condition (1) can be replaced by the condition

$$\sup_x |q(x)| \exp(\varepsilon \sqrt{x}) < \infty, \qquad \varepsilon > 0. \tag{2}$$

In the present note, analogous questions are considered for the equation

$$-y'' = k^2 C(x) y - kQ(x) y, \qquad x \geq 0, \tag{3}$$

where $C(x)$ and $Q(x)$ are square matrices of order n, and the eigenvalues of the matrix $C(x)$ are positive and distinct.

It is proved in the paper that if

$$\|C'(x)\| \leq C_1 \exp(-\varepsilon_1 \sqrt{x}), \qquad \|Q(x)\| \leq C_2 \exp(-\varepsilon_2 \sqrt{x}), \tag{4}$$

where $C_1, C_2, \varepsilon_1, \varepsilon_2 > 0$, then the number of eigenvalues of the operator L is finite (the operator L is defined below).

2. We formulate the basic conditions on equation (3). The matrices $C(x)$ and $Q(x)$ satisfy the following conditions:

1°. $\lim_{x\to\infty} C(x) = C(+\infty)$ exists, is finite and nondegenerate.

2°. $\|C'(x)\|^2 + \|C''(x)\| \in L_1 = L_1(0,\infty)$.

3°. $\lim_{x\to\infty} \|C'(x)\| = 0, \qquad \|C'(x)\| \in L_1$.

4°. $\|Q(x)\| \in L_1$.

Lemma. If conditions 1°–4° are fulfilled, then equation (3) has $2n$ linearly independent solutions $y_1, y_2, \dots, y_{2n}$. These solutions, for any fixed $x \geq x_0$, are regular for $\operatorname{Im} k > 0$ and continuous for $\operatorname{Im} k \geq 0$. The solutions $y_1, y_2, \dots, y_n \in L_2$, while the solutions $y_{n+1}, \dots, y_{2n}$, and no nontrivial combination of them, belong to L_2.

Introduce the operator

$$l = -d^2/dx^2 - k^2 C(x) + kQ(x).$$

Let D be the collection of all vector-functions $y \in L_2$, all components of which are absolutely continuous on every finite interval $[0,a]$, $a > 0$, and such that $ly \in L_2$. Denote by D_L the collection of all vector-functions $y \in D$ such that

$$Ay(0,k) + By'(0,k) = 0, \tag{5}$$

where A and B are constant square matrices of order n, for which one of the following conditions is satisfied:

1) if $B = 0$, then $\det A \neq 0$.

2) if $B \neq 0$, then also $\det B \neq 0$.

By L we denote the operator in L_2 with domain D_L and $Ly = ly$ for $y \in D_L$.

3. Introduce the function

$$D(k) = \det\big[(AY(x) + BY'_x(x))|_{x=0}\big], \tag{6}$$

where $Y(x) = (y_1, y_2, \dots, y_n)$.

Following B. S. Pavlov ([2]), we shall call the point $k = k_0$ a singular point of the operator L if $D(k_0) = 0$. The multiplicity of the singular point $k = k_0$ will be called the multiplicity of the root $D(k_0) = 0$. The set of all singular points of the operator L will be denoted by E, the set of all eigenvalues by E_0, the set of all singular points lying on the real axis $(-\infty,\infty)$ by E_1, the set of singular points of infinite multiplicity by E_2, and the set of all accumulation points of eigenvalues by E_3.

Theorem 1. Suppose that conditions 1^0–4^0 are satisfied. Then:

1) The set of eigenvalues, counted with multiplicities, satisfies the condition

$$\sum |\operatorname{Im} k_0| < \infty.$$

2) $E_3 \subset E_1$.

3) The set E_1 is bounded, closed, has measure zero, and satisfies the condition

$$\sum |l_k| \ln |l_k| > -\infty,$$

where $|l_k|$ is the length of the interval of adjacency l_k to the set E_1, and the summation extends over all bounded intervals of adjacency.

Theorem 2. Suppose that condition 1^0–4^0 is satisfied. In addition, let

$$\|C'(x)\| \leq C_1 \exp(-\varepsilon_1 x^\alpha), \qquad \|Q(x)\| \leq C_2 \exp(-\varepsilon_2 x^\alpha),$$

where $C_1, C_2, \varepsilon_1, \varepsilon_2 > 0,\ 0 < \alpha < 1/2$.

Then:

1) $E_1 \subset E_2$.

2) The set E_2 is bounded, closed, has measure zero, and satisfies the condition

$$\sum |l_k|^{(1-2\alpha)/(1-\alpha)} < \infty,$$

where $|l_k|$ is the length of the interval of adjacency l_k to the set E_2, and the summation extends over all bounded intervals of adjacency.

Theorem 3. If all the conditions of Theorem 2 are satisfied and $\alpha = 1/2$, then the number of eigenvalues of the operator L is finite.

The methods of proof are analogous to the methods used in papers ([2],[3]).

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REFERENCES

[1] M. A. Naimark, *Trudy Moskov. matem. obshch.*, **3**, 181 (1954).
[2] B. S. Pavlov, *Problems of Mathematical Physics*, issue 1, Leningrad, 1966.
[3] M. V. Fedoryuk, *DAN*, **169**, No. 2, 288 (1966).

Note: Figure translations are in progress. See original paper for figures.