CONTACT INVERSE PROBLEMS OF GENERALIZED MAGNETIC POTENTIALS

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Abstract

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CONTACT INVERSE PROBLEMS OF GENERALIZED MAGNETIC POTENTIALS

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We consider the problem of determining the shape of a contact body from the values of the external generalized magnetic potential, which is represented as the sum of the potential of volume masses and of a simple layer. This problem is a typical example of ill-posed problems, methods for solving which have been developed in works ([1,3,5,10,11]) and others. Thus, in ([11]), using the example of the logarithmic potential of volume masses of constant density, an effective method was proposed for solving the external contact problem by reducing this problem to an equivalent nonlinear equation of the first kind. The question of uniqueness of the solution of the external contact problem was studied in ([1,6,9,12]) for harmonic potentials of constant densities under a number of restrictions on the boundary of the bodies. In the author' s paper ([7]), the uniqueness of the solution of the indicated problem was studied for harmonic and metaharmonic potentials of volume masses of variable densities.

In the present article the uniqueness of the solution of the indicated problem is studied for generalized magnetic potentials of a general linear elliptic operator of the second order with variable coefficients, and arbitrary contact bodies with variable densities are considered. We note that in the case of the Laplace operator the results obtained are new also for harmonic potentials ($n \geqslant 2$).

1°. Consider the uniformly elliptic operator

$$Lu = \sum_{i,k=1}^{n} a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{k=1}^{n} b_k \frac{\partial u}{\partial x_k} + cu, \tag{1}$$

where

$$a_{ik} = a_{ki} \in C^{(2,\lambda)}(D_0'), \qquad b_k \in C^{(1,\lambda)}(D_0'), \qquad c \in C^{(0,\lambda)}(D_0') \quad (0 < \lambda < 1)$$

are functions of the point $x = (x_1, \dots, x_n) \in D'_0$; D'_0 is a bounded domain of Euclidean space E^n ($n \geqslant 2$). Suppose, moreover, that the coefficients of the operator (1) satisfy the condition

$$c(x) \leqslant 0, \qquad c^*(x) \leqslant 0 \quad \text{for } x \in D'_0, \tag{2}$$

where c^* is the coefficient of the function v in the operator L^*v (L^* is the operator adjoint to L in the sense of Lagrange), i.e.

$$c^*(x) = c - \sum_{k=1}^{n} \frac{\partial e_k}{\partial x_k}, \qquad e_k = b_k - \sum_{i=1}^{n} \frac{\partial a_{ki}}{\partial x_i}.$$

Let A_α ($\alpha = 1, 2$) be open sets with boundaries Γ_α; A_α is the union of a finite number of domains T^j_α with boundaries S^j_α of class $A^{(1,\lambda)}$ (see ([7])); $\overline{A}_\alpha \subset D_0$, D_0 is an arbitrary domain, $\overline{D}_0 \subset D'_0$. We introduce generalized potentials of volume masses and generalized potentials of sim-

of the simple layer

$$U^\alpha(x) = U(x; A_\alpha, z_\alpha) = \int_{A_\alpha} \Omega(x, y) z_\alpha(y)\, dy; \tag{3}$$

$$V^\alpha(x) = V(x; \Gamma_\alpha, \xi_\alpha) = \int_{\Gamma_\alpha} \Omega(x, y) \xi_\alpha(y)\, d_{yS}, \tag{4}$$

where $\Omega(x, y)$ is the principal elementary solution (see (2)); moreover, if $z_\alpha \in C^{(0,\lambda)}(\bar{A}_\alpha)$, then

$$L[U^\alpha(x)] = \begin{cases} -z_\alpha(x), & x \in A_\alpha, \\ 0, & x \in D_0 \setminus \bar{A}_\alpha; \end{cases}$$

$z_\alpha(y) \neq 0$ ($\xi_\alpha(y) \neq 0$) almost everywhere for $y \in A_\alpha$ ($y \in \Gamma_\alpha$).

The function

$$Z^\alpha(x) = Z(x; A_\alpha, z_\alpha; \Gamma_\alpha, \xi_\alpha) = \beta U^\alpha(x) + \gamma V^\alpha(x), \tag{5}$$

where β, γ are real numbers, $\beta^2 + \gamma^2 \neq 0$, will be called a **generalized magnetic potential** (see [4], the case $\beta = \gamma = 1$, L is the Laplace operator, $n = 3$).

In the case where L is the Laplace operator, the functions (3), (4), and (5) will be called, respectively, **harmonic potentials** ($n \geq 2$) of volume masses, of a simple layer, and harmonic potentials.

2°. We investigate uniqueness of the solution of the indicated problem for generalized magnetic potentials of variable densities of constant sign. Denote

$$B = (A_1 \cup A_2) \setminus \bar{A}_0, \qquad \bar{A}_0 = \bar{A}_1 \cap \bar{A}_2. \tag{6}$$

Theorem 1. If, for the set A_α ($\alpha = 1, 2$) of nonnegative functions $z_\alpha \in C^{(1)}(\bar{A}_\alpha)$, $\xi_\alpha \in C^{(0)}(\Gamma_\alpha)$, and generalized magnetic potentials $Z^\alpha(x)$ ($\alpha = 1, 2$), defined by formula (5) (where β, γ are nonnegative numbers), the condition

$$Z^1(x) = Z^2(x) \quad \text{for } x \in D_0 \setminus \bar{B},$$

is satisfied, then

$$A_1 = A_2$$

and, moreover, if $\beta \neq 0$, then $z_1(x) = z_2(x)$, $x \in A_1$; if $\gamma \neq 0$, then $\xi_1(x) = \xi_2(x)$, $x \in \Gamma_1$.

We shall call the sets A_1 and A_2 **externally contact** if every component of the set A_0 ($A_0 \neq 0$) has a part Γ_* ($\operatorname{mes} \Gamma_* \neq 0$) of the common boundary of the $(n-1)$-dimensional space with one of the components of the set $D_0 \setminus (\bar{A}_1 \cup \bar{A}_2)$ (see [7]).

Theorem 2. If, for externally contact sets A_α ($\alpha = 1, 2$), nonnegative functions $z \in C^{(1)}(\bar{A}_\alpha)$, $\xi \in C^{(0)}(\Gamma_\alpha)$, and also nonnegative numbers β and γ, equality of the external generalized magnetic potentials holds, i.e.,

$$Z(x; A_1, z; \Gamma_1, \xi) = Z(x; A_2, z; \Gamma_2, \xi) \quad \text{for } x \in D_0 \setminus (\bar{A}_1 \cup \bar{A}_2),$$

then

$$A_1 = A_2.$$

3°. We investigate uniqueness of the solution of the posed problem for generalized potentials of volume masses of variable densities not of constant sign. Let the coefficients of the operator (1) satisfy condition (2) and, in addition,

$$\frac{\partial}{\partial y_n} L^* h(y) = L^* \frac{\partial}{\partial y_n} h(y), \qquad y = (y_1, \dots, y_n).$$

Let the function $z(y) \in C^{(1)}(D_0')$ satisfy the condition

$$z(y) = \delta(y)\eta(y), \tag{7}$$

where

a) $\eta(y) \in C^{(1)}(D_0')$ is a nonnegative function, monotone with respect to y_n,

$$ \tag{8} $$

b) the function $\delta(y)$, generally speaking not of constant sign, satisfies the condition

$$\partial\delta/\partial y_n = 0. \tag{9}$$

Theorem 3. *If, for the external contact sets A_α $(\alpha = 1, 2)$ of generalized potentials of volume masses $U(x; A_\alpha, z)$ with density z of class (7)–(9), the equality*

$$U(x; A_1, z) = U(x; A_2, z) \quad \text{for } x \in D_0 \setminus (\overline{A}_1 \cup \overline{A}_2),$$

holds, then

$$A_1 = A_2.$$

Let (ρ, θ) denote the spherical coordinates of a point y of the space E^n $(n \geq 2)$.

Let the function $z(y)$ satisfy the conditions

$$z(y) = \delta(y)\eta(y), \tag{10}$$

where

a) the function $\delta(y)$, generally speaking not of constant sign, satisfies the condition

$$\partial\delta/\partial\rho = 0; \tag{11}$$

b) $\rho^n\eta(y) = \psi(y)$ is a function monotone with respect to ρ.

$$ \tag{12} $$

Theorem 4. *If, for the external contact sets A_α $(\alpha = 1, 2)$ of harmonic potentials of volume masses $U(x; A_\alpha, z)$ $(n \geq 2)$ with density $z(y)$ of class (10) –(12), the equality*

$$U(x; A_1, z) = U(x; A_2, z) \quad \text{for } x \in D_0 \setminus (\overline{A}_1 \cup \overline{A}_2)$$

holds, then

$$A_1 = A_2.$$

4°. We investigate the uniqueness of the solution of the external contact inverse problem for generalized potentials of a simple layer of variable densities, generally speaking not of constant sign.

Let the coefficients of operator (1) satisfy condition (2).

Theorem 5. *If, for the external contact sets A_α $(\alpha = 1, 2)$ of generalized potentials of a simple layer $V(x; \Gamma_\alpha, \zeta)$ with density $\zeta \in C^{(0)}(\Gamma_\alpha)$, generally speaking of variable sign, the equality*

$$V(x; \Gamma_1, \zeta) = V(x; \Gamma_2, \zeta) \quad \text{for } x \in D_0 \setminus (\overline{A}_1 \cup \overline{A}_2),$$

holds, then

$$\Gamma_1 = \Gamma_2.$$

We note that if, in the conditions of Theorem 5, one does not assume the external contactness of the bodies A_α, then, even in the case of constant density, in work (8) an example is given of two different star-shaped bodies having equal external logarithmic potentials of a simple layer.

For generalized potentials of a simple layer of operator (1), the coefficients of which satisfy condition (2), the following theorems hold.

Theorem 6. *Let A_α be open sets consisting of a finite number of convex domains. If the external generalized potentials of a simple layer $V(x; \Gamma_\alpha, 1)$ of constant density equal to unity coincide, i.e.*

$$V(x; \Gamma_1, 1) = V(x; \Gamma_2, 1) \quad \text{for } x \in D_0 \setminus (\overline{A}_1 \cup \overline{A}_2),$$

$$\Gamma_1 = \Gamma_2.$$

Introduce the notation (see (7)). Γ^l is the boundary of the set $\bar{A}_1 \cup \bar{A}_2$. If $T_1^j \neq T_2^k$, then we denote

$$\Gamma_1^i = \Gamma_1 \cap (\bar{A}_1 \cap \bar{A}_2), \qquad \Gamma_1^l = \Gamma_1 \setminus \Gamma_1^i,$$
$$\Gamma_2^l = \Gamma_2 \cap \Gamma^i, \qquad \Gamma_2^i = \Gamma_2 \setminus \Gamma_2^l.$$

If $T_1^i = T_2^k$, then we put $S_1^j = (S_1^j)^l$, $S_2^k = (S_2^k)^l$.

Theorem 7. Let the bounded sets A_α with boundaries Γ_α and the functions $\zeta_\alpha(y) \in C^{(0)}(\Gamma_\alpha)$ satisfy the condition

$$\int_{\Gamma_1^i} |\zeta_1(y)|\, d_y S + \int_{\Gamma_2^i} |\zeta_2(y)|\, d_y S < \int_{\Gamma_1^l} |\zeta_1(y)|\, d_y S + \int_{\Gamma_2^l} |\zeta_2(y)|\, d_y S.$$

If $A_1 \neq A_2$, then the exterior generalized simple-layer potentials $V(x; \Gamma_1, \zeta_1)$ and $V(x; \Gamma_2, \zeta_2)$ do not coincide, i.e., if $A_1 \neq A_2$, then there exists a point $\hat{x} \in D_0 \setminus (\bar{A}_1 \cup \bar{A}_2)$ such that

$$V(\hat{x}; \Gamma_1, \zeta_1) \neq V(\hat{x}; \Gamma_2, \zeta_2).$$

Remark 1. Under the hypotheses of Theorems 1-7, the restrictions imposed on the smoothness of the boundaries Γ_α, as well as of the functions z_α, ζ_α, can be weakened (see (7)).

Remark 2. If ΓD_0, the boundary of the domain D_0, belongs to the class $A^{(1,\lambda)}$ and the set $D_0 \setminus (\bar{A}_1 \cup \bar{A}_2)$ consists of one component, then the assertions of Theorems 2-6 remain valid if everywhere in the hypotheses of these theorems, instead of the assumption $x \in D_0 \setminus (\bar{A}_1 \cup \bar{A}_2)$, we write $x \in \Gamma D_0$.

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Note: Figure translations are in progress. See original paper for figures.