ON DIFFERENTIAL EQUATIONS WITH UNBOUNDED OPERATORS GENERATING NONANALYTIC SEMIGROUPS

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Abstract

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ON DIFFERENTIAL EQUATIONS WITH UNBOUNDED OPERATORS GENERATING NONANALYTIC SEMIGROUPS

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In a Banach space E one considers the problem

$$v'(t) + Av(t) = f(t) \quad (0 < t \leq t_0), \qquad v(0) = v_0. \tag{1}$$

Here A is a linear operator having an everywhere dense domain of definition D. A function $v(t)$, continuous on $[0, t_0]$, is called a solution of problem (1) if it satisfies (1) and the functions $v'(t)$ and $Av(t)$ are continuous on $(0, t_0]$.

It is assumed that there exists a continuously differentiable operator-function $T(t) = \exp\{-tA\}$ $(t > 0)$—the resolving operator of problem (1)—satisfying the relation $T'(t) = -AT(t)$ and commuting with A. It is further assumed that there exists an integer $n \geq 0$ such that $T(t)v_0 \to v_0$ as $t \to +0$ for every $v_0 \in D(A^n)$. Then, if $v(t)$ is a solution of problem (1) and the function $A^n v(t)$ is continuous on $(0, t_0]$, then

$$v(t) = T(t)v_0 + \int_0^t T(t-s)f(s)\,ds = T(t)v_0 + Q(t)f. \tag{2}$$

Finally, it is assumed that the operator-functions $T(t)$ and $AT(t)$ have power-type singularities at zero. Under these conditions it is shown that formula (2) determines a solution of problem (1), provided v_0 and $f(t)$ are sufficiently smooth. Moreover, if $\|T(t)\|_{E \to E}$ is summable, then it suffices to require of $f(t)$ that it satisfy a certain Hölder condition (Sec. 1). This fact is established by means of the method of fractional integration by parts (see (^{1})).

If $T(t)$ has a singularity as $t \to +0$, then problem (1) is not well posed. In Sec. 2, from $T(t)$ a pair of Banach spaces E^+ and E^-, $E^+ \subset E \subset E^-$, is constructed, in which problem (1) becomes well posed. It follows from this that it is solvable in these spaces under the assumption only of Hölder continuity of $f(t)$, for any polarity of $AT(t)$.

The results of the first part of the work develop the investigations of the paper (^{2}). In its second part the problem

$$v'(t) + A(t)v(t) = f(t) \quad (0 < t \leq t_0), \qquad v(0) = v_0 \tag{3}$$

is studied, with a variable operator $A(t)$ having a constant everywhere dense domain of definition D. The resolving operator $U(t, \tau)$ of problem (3) can be sought by the method of "frozen coefficients" as a solution of the integral equations

$$U(t, \tau) = T_\tau(t - \tau) + \int_\tau^t U(t, s)[A(\tau) - A(s)]T_\tau(s - \tau)\, ds, \tag{4}$$

$$U(t, \tau) = T_t(t - \tau) + \int_\tau^t T_t(t - s)[A(t) - A(s)]U(s, \tau)\, ds. \tag{5}$$

Here $T_\tau(t) = \exp\{-tA(\tau)\}$ is the resolving operator of problem (1) with $A = A(\tau)$. If $T_\tau(t)$ is strongly continuous in t as $t \to +0$, while the norm

$\|AT_\tau(t)\|_{E\to E}$ behaves like $t^{-\beta}$ for $\beta < 2$, then equation (4) has a unique strongly continuous solution $U(t, \tau)$ jointly in t and τ, under the assumption of a certain Hölder continuity of the operator-valued function $A(t)A^{-1}(0)$. The operator-valued function $U(t, \tau)$ will be the resolving operator for problem (3) if the adjoint operator-valued function $A^*(t)$ has the same smoothness. This was established in (^{3}). In (^{4}) the result is generalized to the case where $T_\tau(t)$ has a power singularity as $t \to +0$. In (^{5}), for $\beta < 7/5$, the resolving operator is constructed under the assumption of strong continuity of $T_\tau(t)$ in t and Hölder continuity only of $A(t)A^{-1}(0)$. Finally, in (^{6}) this result is generalized to the case where $T_\tau(t)$ has a singularity as $t \to +0$.

In Sec. 3 the resolving operator is constructed for $\beta < 2$, under the assumption of Hölder continuity only of $A(t)A^{-1}(0)$. We note that the condition $\beta < 2$ is necessary for the applicability of the method of "frozen coefficients."

1. Suppose that, for some $C, \delta, \alpha, \beta > 0$, $\beta \geq 1 + \alpha$,

$$\|T(t)\|_{E\to E} \leq ce^{-\delta t}t^{-\alpha}, \qquad \|AT(t)\|_{E\to E} \leq ce^{-\delta t}t^{-\beta}. \tag{6}$$

Denote by $C^\gamma = C^\gamma([0, t_0], E)$ $(0 \leq \gamma \leq 1)$ the closure of the set K of all polynomials with coefficients in $\bigcap_{n=1}^\infty D(A^n)$ in the norm

$$\|f\|_{C^\gamma} = \max_{0\leq t\leq t_0} \|f(t)\|_E + \sup_{0\leq t<t+\Delta t\leq t_0} \Delta t^{-\gamma}\|f(t + \Delta t) - f(t)\|_E.$$

The closure of K in the norm $\|f\|_{C^0} + \|f^{(n)}\|_{C^\gamma} = \|f\|_{C^{\gamma+n}}$, for an integer $n \geq 0$, forms the Banach space $C^{\gamma+n}$. Since $D = D(A)$ is everywhere dense, $C^{\gamma+n}$ does not depend on A. The totality of all such functions $f(t) \in C^{\gamma+n}$ $(n \geq 1)$ for which $f^{(i)}(0) \in D(A^{n-i})$ for $i = 0, \dots, n-1$, forms the subspace $C^{\gamma+n}(A)$. By definition we set $C^{\gamma+0}(A) = C^{\gamma}$.

Theorem 1. Let $n < \alpha < n+1$, $f(t) \in C^{\gamma+n}(A)$, $v_0 \in D(A^{n+1})$, for some integer $n \geq 0$ and some γ in $((\beta - n - 1)/(\beta - \alpha), 1)$. Then formula (2) defines a solution of problem (3), and the estimates

$$\|v'(t)\|_E \leq ct^{n-\alpha}, \qquad \|Av(t)\|_E \leq ct^{n-\alpha}$$

are valid.

We shall carry out the proof for the case $n = 0$. Since it is obvious that the function $Q(t)f$ is a solution of equation (1), if $f(t) \in K$, for such $f(t)$ it suffices to establish the estimate

$$\|AQ(t)f\|_E \leq ct^{-\alpha}\|f\|_{C^\gamma}. \tag{7}$$

Represent the function $AQ(t)f$ in the form of a Stieltjes integral

$$AQ(t)f = \int_0^t AT(t-\tau)f(\tau)\,d\tau = \int_0^t d_\tau[T(t-\tau)]f(\tau) = \int_0^1 d_s[\Phi(s)]\varphi(s).$$

Here $\Phi(s) = T(t - ts)$, $\varphi(s) = f(ts)$. Further, the last integral can be represented as the limit of sums

$$s_n = \sum_{k=1}^{2^n} [\Phi(k/2^n) - \Phi((k-1)/2^n)]\,\varphi(k/2^n).$$

Since estimates (6) are satisfied, we have $\|\Phi(s + \Delta s) - \Phi(s)\|_{E \to E} \leq ct^{\delta(1-\beta+\alpha)-\alpha}\Delta s^\delta$ for any δ in $[0, 1]$. Since $f(t) \in C^\gamma$, we have $\|\varphi(s + \Delta s) - \varphi(s)\|_E \leq ct^\gamma \Delta s^\gamma$. The last two estimates allow one to establish that, for $1 - \gamma < \delta < (1 - \alpha)/(\beta - \alpha)$, the estimate

$$\|s_{n+1} - s_n\|_E \leq c(1-\alpha-\delta\beta+\delta\alpha)^{-1} t^{\delta(1-\beta+\alpha)+\gamma-\alpha}\, 2^{-n(\delta+\gamma-1)}\|f\|_{C^\gamma}, \qquad n = 0, 1, \dots$$

is valid.

Finally, since $\|s_0\|_E \leq ct^{-\alpha}\|f\|_{C^\gamma}$, for $AQ(t)f = s = \lim s_n$ estimate (7) is valid. In the case of arbitrary n, one must first transform formula (2) by integration by parts.

2. Denote by $E^+(A)$ the closure of the set $D(A^{n+1})$ in the norm

$$\|v\|_{E^+} = \sup_{t \geq 0} \|T(t)v\|_E.$$

It is clear that $\|v\|_E \leq \|v\|_{E^+}$. Further,

$$\|v\|_{E^+} \leq c\|A^{n+1}v\|_E.$$

For the proof one may use the identity

$$A^{-n-1} = \int_0^\infty \cdots \int_0^\infty T(s_1 + \cdots + s_{n+1})\, ds_1 \cdots ds_{n+1}.$$

The norms of the spaces E, $E^+(A)$, and $D(A^{n+1})$ are compatible; from the fundamentalness of a sequence in the strong norm and its convergence to zero in the weak norm there follows convergence to zero in the strong norm. Therefore the embeddings

$$D(A^{n+1}) \subset E^+(A) \subset E$$

hold.

Denote by $E^-(A)$ the closure of the set E in the norm

$$\|v\|_{E^-} = \sup_{t > 0} t^\alpha \|T(t)v\|_E.$$

Analogously it is established that the embeddings

$$E \subset E^-(A) \subset D(A^{-n-1})$$

hold. Here by $D(A^{-n-1})$ is meant the closure of the set E in the norm $\|A^{-n-1}v\|_E$.

Theorem 2. The operator-function $T(t)$ is a uniformly bounded and strongly continuous semigroup in the spaces $E^\pm(A)$. The estimate

$$\|AT(t)\|_{E^\pm \to E^\pm} \leq ct^{-\beta}$$

is valid.

This theorem makes it possible to investigate problem (1) in the spaces $E^\pm(A)$, i.e. to study solutions with increased smoothness and generalized solutions of this problem. Problem (1) is well posed in the spaces $E^\pm(A)$. The space of well-posedness for problem (1) can also be constructed by other methods, for example, with the aid of the norm

$$\|v\|_{E^p} = \left(\int_0^\infty \|T(t)v\|_E^p \, dt \right)^{1/p}, \qquad 1 \leq p < +\infty.$$

Such a device was used earlier in the study of partial differential equations (see (^{4})).

3. We turn to the consideration of problem (3). Let $T_\tau(t)$, for each τ from $[0, t_0]$, satisfy the estimates (6) with constants independent of τ. Suppose, moreover, that
$$\|A^2(\tau)T_\tau(t)\|_{E\to E} \le ce^{-\delta t}t^{-\eta}. \tag{8}$$

Theorem 3. Let $1 < \beta < 2$, $0 \le \alpha \le \beta - 1$, $2\beta - \alpha < \eta < 2 + \beta - \alpha$. Suppose that for some ε from $(\eta - 2\beta + \alpha,\ 2 + \beta)$ the inequality

$$\|[A(t) - A(\tau)]A^{-1}(0)\|_{E\to E} \le c|t-\tau|^{\beta-1+\varepsilon}$$

is fulfilled.

Then there exists an operator-function $U(t,\tau)$, defined for all $0 \le \tau < t \le t_0$, continuous in the totality of variables, continuously differentiable with respect to t, and satisfying the relations

$$U(t,\tau) = U(t,s)U(s,\tau) \quad (0 \le \tau < s < t \le t_0), \qquad \frac{\partial}{\partial t}[U(t,\tau)] = -A(t)U(t,\tau). \tag{9}$$

The estimates

$$\|U(t,\tau)\|_{E\to E} \le c|t-\tau|^{-\alpha}, \qquad \|A(t)U(t,\tau)A^{-1}(\tau)\|_{E\to E} \le c|t-\tau|^{-\alpha},$$
$$\|A(t)U(t,\tau)\|_{E\to E} \le c|t-\tau|^{-\beta} \tag{10}$$

are valid.

The operator-function $U(t,\tau)A^{-1}(0)$ tends strongly and uniformly in t and τ to the operator $A^{-1}(0)$ as $t - \tau \to +0$. For $\tau < t$ it is continuously differentiable with respect to τ and satisfies the relation

$$\frac{\partial}{\partial \tau}[U(t,\tau)A^{-1}(0)] = U(t,\tau)A(\tau)A^{-1}(0). \tag{11}$$

Suppose $f(t) \in C^\gamma$ for some γ from $((\beta-1)/(\beta-\alpha),\ 1)$ and $v_0 \in D$. Then there exists a unique continuous on $[0, t_0]$ and continuously diff-

differentiable on $(0, t_0]$ solution $v(t)$ of problem (3), the formula

$$v(t) = U(t,0)v_0 + \int_0^t U(t,s)f(s)\,ds = U(t,0)v_0 + Q(t,0)f, \tag{12}$$

is valid, and the estimates

$$\|v'(t)\|_E \le ct^{-\alpha}, \qquad \|A(t)v(t)\|_E \le ct^{-\alpha}$$

hold.

We give the scheme of the proof. The interval $[0, t_0]$ is divided into n equal parts by the points $\tau_0 = 0, \tau_1, \ldots, \tau_n = t_0$. The operators $U_n(t, \tau)$ are defined, for arbitrary $\tau_{i-1} \leq \tau < \tau_i \leq \cdots \leq t = \tau_k,\ \tau_{r-1} \leq \tau_r \leq \tau_r$, by the formula

$$U_n(t, \tau) = T_{\tau_k}(\tau_k - \tau_{k-1}) \ldots T_{\tau_i}(\tau_i - \tau). \tag{13}$$

It is shown that the operators $U_n(t, \tau)$ satisfy an identity $(*)$ of Volterra type, analogous to relation (4) for the operator $U(t, \tau)$. In this identity, sums occur instead of integrals. With the aid of (4) and $(*)$ it is established that $U_n(t, \tau) \to U(t, \tau)$, the solution of (4). Next, the operators $\widetilde{U}_n(t, \tau)$ are constructed analogously to the operators $U_n(t, \tau)$, only now the points τ must coincide with the points of the subdivision of $[0, t_0]$. The operators $\widetilde{W}_n(t, \tau) = A(t)\widetilde{U}_n(t, \tau)A^{-1}(\tau)$ are introduced and it is shown that $\widetilde{W}_n(t, \tau) \to W(t, \tau)$. Here $W(t, \tau)$ denotes the solution of the equation into which equation (5) is transformed under the substitution $W(t, \tau) = A(t)U(t, \tau)A^{-1}(\tau)$. Since $U_n(t, \tau) - \widetilde{U}_n(t, \tau) \to 0$, it follows from this that $U(t, \tau) = \lim U_n(t, \tau)$ and $A(t)U(t, \tau)A^{-1}(\tau)$ are jointly continuous in the variables for $t > \tau$ and satisfy the first two estimates (10). This is the first main fact. Since $U_n(t, \tau)$ is constructed in the form of the product (13), $U(t, \tau)$ satisfies the first of relations (9). This is the second main fact. Next one uses the identity following from (4), (5), and (9) $(0 \leq \tau < t < t + \Delta t \leq t_0,\ v_0 \in D)$

$$[U(t+\Delta t, \tau) - U(t, \tau)]v_0 = \int_t^{t+\Delta t} U(t+\Delta t, s)[A(t) - A(s)]\left\{T_t(s-\tau)v_0 + \right.$$

$$\left. + \int_\tau^t T_t(s-z)[A(t) - A(z)]U(z, \tau)v_0\, dz \right\} ds + [T_t(t+\Delta t-\tau) - T_t(t-\tau)]v_0 +$$

$$+ \int_\tau^t [T_t(t+\Delta t - s) - T_t(t-s)][A(t) - A(s)]U(s, \tau)v_0\, ds,$$

by means of which the derivative

$$\frac{\partial}{\partial t}[U(t, \tau)v_0]$$

is computed.

For the proof of the second part of the theorem it is necessary to represent the function $A(t)Q(t, 0)f$ in the form of a Stieltjes integral (see (11))

$$A(t)Q(t,0)f = \int_0^t A(t)U(t,s)f(s)\,ds = \int_0^t d_s[A(t)U(t,s)]A^{-1}(s)f(s)$$

and, for its estimate, to use the method of Section 1.

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