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Abstract

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A NON-SELF-ADJOINT ONE-DIMENSIONAL PERTURBATION OF THE OPERATOR OF MULTIPLICATION BY THE INDEPENDENT VARIABLE

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In many cases, smallness of the norm of a perturbation ensures the similarity of the perturbed and unperturbed operators (see, for example, ([1,2])). In the present note we do not assume that the perturbation is small in norm; however, we impose on it certain smoothness conditions. This gives rise to a spectral theory with sufficiently simple regularities (although more complicated than in the self-adjoint case). For simplicity we restrict ourselves to the case of a one-dimensional perturbation of the operator of multiplication by the independent variable, since even this case already makes it possible to observe effects of substantial interest. The situation arising here is in many respects analogous to the theory of non-self-adjoint differential or difference operators with spectral singularities (see ([3,4])). In particular, the spectral expansion is expressed by means of regularized values of divergent integrals. It is interesting to note that the perturbed operator may have eigenvalues for which Fredholm' s second theorem is not valid (i.e., eigenvalues with nonzero index). Nevertheless, even in this case it is possible to construct the corresponding Parseval equality and to prove a theorem on the completeness of the eigenfunctions.

1. We shall study an operator of the form $T = S + V$, where S is the operator of multiplication by the independent variable in the Hilbert space $H = L_2(-\infty, \infty)$, i.e. $Sf(x) = xf(x)$, $D(S) = \{f : f(x), xf(x) \in H\}$, and V is a one-dimensional operator on which we shall impose the restriction described below.

Let Φ be the linear manifold of those $\varphi \in H$ that admit a continuation (denoted below also by φ) from the real x-axis into the complex $z = x + iy$-plane, holomorphic in some strip $|\operatorname{Im} z| < \varepsilon$, and satisfying the condition

$$|\varphi|_\eta^2 \overset{df}{=} \sup_{|y|\le\eta} \int_{-\infty}^{\infty} |\varphi(x+iy)|^2\, dx < \infty \tag{1}$$

for all $\eta < \varepsilon$. We shall assume that $Vf = (f, \beta)\alpha$, where α and β are certain fixed elements of Φ. From this assumption it follows that the function

$$\chi(\lambda) \overset{df}{=} 1 + \int_{-\infty}^{\infty} \frac{\alpha(x)\overline{\beta(x)}\,dx}{x - \lambda}, \qquad \operatorname{Im}\lambda \neq 0, \tag{2}$$

admits holomorphic continuations $\chi_+(\lambda)$ $(\chi_-(\lambda))$ from the half-plane $\operatorname{Im}\lambda > 0$ $(\operatorname{Im}\lambda < 0)$ into the half-plane $\operatorname{Im}\lambda > -\varepsilon$ $(\operatorname{Im}\lambda < +\varepsilon)$, and moreover

$$\chi_+(\lambda) = 1 + o(1) \qquad (\chi_-(\lambda) = 1 + o(1)), \qquad |\lambda| \to \infty \tag{3}$$

uniformly in each region $\operatorname{Im}\lambda \geq -\eta$ $(\operatorname{Im}\lambda \leq +\eta)$, $\eta < \varepsilon$.

2. For each $\varphi \in H$ put

$$\varphi_\lambda(x) = (x - \lambda)^{-1}\varphi(x), \qquad \operatorname{Im}\lambda \neq 0. \tag{4}$$

It is easy to see that $a_\lambda \in D(T)$ and

$$Ta_\lambda = \lambda a_\lambda + \chi(\lambda)a, \qquad \operatorname{Im}\lambda \neq 0. \tag{5}$$

From this it is easy to derive the following: in order that a nonreal λ_k be an eigenvalue of T of (geometric) multiplicity m_k, it is necessary and sufficient that λ_k be a root of $\chi(\lambda)$ of multiplicity m_k; moreover $a_{\lambda_k}^{(0)}, \dots, a_{\lambda_k}^{(m_k-1)}$, where $a_\lambda^{(j)} \overset{df}{=} \left(\dfrac{d}{d\lambda}\right)^j a_\lambda$, is a chain of root functions of T corresponding to the eigenvalue λ_k. In passing from T to T^*, in the preceding statement one should replace $\chi(\lambda)$ by $\overline{\chi(\overline{\lambda})}$ and α by β.

3. The real roots σ_k of the function $\chi_+(\lambda)\chi_-(\lambda)$ will be called spectral singularities of T. Let us describe the root functions of T corresponding to σ_k. For this purpose, for each $\varphi \in H$ and each real σ denote by $\varphi_{\sigma+}^{(j)}$ $(\varphi_{\sigma-}^{(j)})$ the functional defined by the relation

$$(f, \varphi_{\sigma\pm}^{(j)}) \overset{df}{=} \lim_{\tau \to \pm 0} \frac{1}{j!} \int_{-\infty}^{\infty} \frac{f(x)\overline{\varphi(x)}\,dx}{[x - (\sigma + i\tau)]^{j+1}} \tag{6}$$

for all those $f \in H$ for which the limit (6) exists. Let σ_k be a spectral singularity of T: denote by n_k^+, n_k^-, p_k, q_k the multiplicity of σ_k as a root of the functions $\chi_+(\sigma)$, $\chi_-(\sigma)$, $\alpha(\sigma)$, $\beta(\sigma)$, respectively. By definition, the root functions of T corresponding to σ_k are the functionals $\alpha_{\sigma_k+}^{(j)}$, $j = 0, \dots, n_k^+ - 1$, and $\alpha_{\sigma_k-}^{(j)}$, $j = 0, \dots, n_k^- - 1$. In passing to T^* one should here replace α by β.

Let $n_k \overset{df}{=} \min(n_k^+, p_k)$, $n_k^* \overset{df}{=} \min(n_k^+, q_k)$. It turns out that also $\min(n_k^-, p_k) = n_k$ and $\min(n_k^-, q_k) = n_k^*$, and that $\alpha^{(j)}_{\sigma_k+} = \alpha^{(j)}_{\sigma_k-} \in H$ for $j = 0, \dots, n_k - 1$, and that $\beta^{(j)}_{\sigma_k+} = \beta^{(j)}_{\sigma_k-} \in H$ for $j = 0, \dots, n_k^* - 1$. Thus the spectral singularity σ_k is an eigenvalue of T of multiplicity n_k and an eigenvalue of T^* of multiplicity n_k^*. In particular, if $n_k = 0$ ($n_k^* = 0$), then σ_k is not an eigenvalue of T (T^*). The operators T and T^* have no eigenvalues other than those described here.

4. Under our assumption on V, the set of eigenvalues and spectral singularities of T is finite.

5. It turns out that the real axis, except for the real eigenvalues, belongs to the continuous spectrum of T. Let us describe the root functions of the continuous spectrum. For this purpose, for arbitrary $f \in H$ and $\tau \neq 0$, put

$$U_\tau f(\sigma) \overset{df}{=} \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x - (\sigma + i\tau)}. \tag{7}$$

As $\tau \to \pm 0$ there exist limits $U_\pm \overset{df}{=} \lim U_\tau$ in the sense of strong convergence of operators in H. If $f \in D(\mathcal{D})$, where $\mathcal{D}$ is the operator of differentiation in H (defined on functions absolutely continuous on every finite interval and square-integrable on the whole axis together with their derivative), the limit

$$(f, U_{\sigma\pm}) \overset{df}{=} \lim_{\tau \to \pm 0} U_\tau f(\sigma) \tag{8}$$

exists uniformly in $\sigma \in (-\infty, \infty)$. It turns out that, for every $f \in D(\mathcal{D})$ and every $\sigma \in (-\infty, \infty)$,

$$\chi_+(\sigma) f(\sigma) - (\beta f, U_{\sigma+})\alpha(\sigma) = \chi_-(\sigma) f(\sigma) - (\beta f, U_{\sigma-})\alpha(\sigma). \tag{9}$$

We denote the common value of the left- and right-hand sides of (9) by (f, b_σ). The functional b_σ is continuous with respect to the norm $[\|f\|^2 + \|\mathcal{D} f\|^2]^{1/2}$ and is a (generalized) eigenfunction of T^* corresponding to the point σ, in the following sense:

$$(Tf, b_\sigma) = \sigma(f, b_\sigma) \tag{10}$$

for all $f \in D(T)$ such that f and $Tf \in D(\mathcal{D})$. We shall denote by a_σ the eigenfunction of T corresponding to the point σ. If, in the formula for b_σ, $\chi_\pm(\sigma)$ is replaced by $\chi_\pm(\overline{\sigma})$, α by β, and β by α, then one obtains the formula for a_σ.

6. It turns out that for any $f, g \in D(\mathcal{D})$ and each real σ that is not a spectral singularity, the limiting values $(R_{\sigma \pm i0} f, g)$, where $R_\lambda = (T - \lambda)^{-1}$, exist, and the following formula for the jump of the resolvent of T is valid:

$$\frac{1}{2\pi i}((R_{\sigma+i0} - R_{\sigma-i0})f, g) = \frac{(f, b_\sigma)\overline{(g, a_\sigma)}}{\chi_+(\sigma)\chi_-(\sigma)}. \tag{11}$$

7. As follows from (11), the density of the spectral function of T on the continuous spectrum, equal to $1/\chi_+(\sigma)\chi_-(\sigma)$, has poles—the spectral singularities σ_k. This complicates the corresponding Parseval equality.

Let $f \in D(\mathcal{D}^\mu)$, where

$$\mu = \overset{df}{\max_{\sigma_k}}(n_k^+ + n_k^- - n_k - n_k^*),$$

and let $\varphi \in \Phi$; then

$$\int_{-\infty}^{\infty} f(x)\overline{\varphi(x)}\,dx = \int_+ (f, b_\sigma)\overline{(\varphi, a_\sigma)}\frac{d\sigma}{\chi_+(\sigma)\chi_-(\sigma)} +$$

$$+\sum_{\sigma_k}\frac{1}{(n_k^- - 1)!}\left\{\left(\frac{d}{d\sigma}\right)^{n_k^- - 1}\frac{(\sigma-\sigma_k)^{n_k^-}}{\chi_-(\sigma)}(f, \beta_{\sigma+})\overline{(\varphi, a_{\sigma-})}\right\}_{\sigma=\sigma_k} +$$

$$+\sum_{\lambda_k}\frac{1}{(m_k - 1)!}\left\{\left(\frac{d}{d\lambda}\right)^{m_k - 1}\frac{(\lambda-\lambda_k)^{m_k}}{\chi(\sigma)}(f, \beta_\lambda)\overline{(\varphi, a_\lambda)}\right\}_{\lambda=\lambda_k}. \tag{12}$$

In the Parseval equality (12), $\int_+$ denotes the regularized value of the integral

$$\int_{-\infty}^{\infty}$$

with respect to the measure $d\sigma/\chi_+(\sigma)\chi_-(\sigma)$, in the sense in which the limit of the right-hand side of (7) as $\tau \to +0$ is the regularized value of the integral

$$\int_{-\infty}^{\infty}(x-\sigma)^{-1}f(x)\,dx$$

(i.e. in the sense of the limiting value from the upper half-plane). In addition, $\sum_{\sigma_k}$ extends over all spectral singularities of T, and differentiation $d/d\sigma$ is to be understood in the sense of (6). The smoothness $f \in D(\mathcal{D}^\mu)$ ensures the existence of the regularized integral in (12)* and the membership of f in the domain of definition of the corresponding functionals $\beta_{\sigma+}^{(j)}$. The sum $\sum_{\lambda_k}$ in (12) extends over all nonreal eigenvalues of T. Equality (12) remains valid if the indices $+$ and $-$ are interchanged everywhere in it.

* We denote the regularized integral in (12) by I_+. I_+ is a bilinear form of $f \in D(\mathcal{D}^\mu)$ and $\varphi \in \Phi$, continuous in the following sense:

$$|I_+| \leq C_\eta \left(\sum_{j=0}^{\mu} \|\mathcal{D}^j f\|^2 \right)^{1/2} |\varphi|_\eta, \qquad 0 < \eta < \varepsilon;$$

here $|\varphi|_\eta$ is the norm defined by relation (1), and C_η is a constant independent of both f and φ.

8. From Parseval' s equality (12) it follows that every function f from the manifold $D(\mathcal{D}^\mu)$, dense in H, is uniquely determined by the following collection of quantities (the T-Fourier transform):

a) the numerical function (f, b_σ), given on the real axis $-\infty < \sigma < \infty$;

b) the vector function

$$\{(f, \beta_{\sigma_k^+}^{(n_k^- - n_k)}), \dots, (f, \beta_{\sigma_k^+}^{(n_k^- - 1)})\},$$

given on the set of real eigenvalues σ_k;

c) the vector function

$$\{(f, \beta_{\lambda_k}^{(0)}), \dots, (f, \beta_{\lambda_k}^{(m_k - 1)})\},$$

given on the set of non-real eigenvalues λ_k.

Let us point out that the vector in (b) consists of n_k, and not n_k^-, components. These components serve as the coefficients in (12) for those $\alpha_{\sigma_k^-}^{(j)}$ which belong to H, i.e., are principal functions of T in the usual sense. As for the quantities

$$(f, \beta_{\sigma_k^+}^{(0)}), \dots, (f, \beta_{\sigma_k^+}^{(n_k^- - n_k - 1)}),$$

also contained in (12), they are expressed by simple formulas in terms of (f, b_σ), and therefore cannot be prescribed arbitrarily.

9. Applying another method of regularizing the integral with respect to the measure $d\sigma / \varkappa_+(\sigma) \varkappa_-(\sigma)$, one can construct Parseval' s equality for arbitrary functions $f \in \widetilde{H}$. For those $f \in H$ which have the following property: for every spectral singularity σ_k the function (see (7))

$$[\varkappa_+(\sigma) f(\sigma) - \alpha(\sigma) U_+ \bar{\beta} f(\sigma)] / (\sigma - \sigma_k)^{n_k^+ + n_k^- - n_k} \tag{13}$$

is integrable with square in a neighborhood of σ_k, Parseval' s equality is written with the aid of the ordinary (unregularized) integral, and the spectral expansion converges in the norm of H.

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Note: Figure translations are in progress. See original paper for figures.