ON THE STABILITY OF BODIES OF REVOLUTION WITH RESPECT TO RADIAL PERTURBATIONS

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Abstract

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1. Among equilibrium figures of revolution, whose gravitational potential inside is $\psi(r)$ (r is the spherical or cylindrical radial coordinate), three are known: a sphere, a cylinder of infinite generatrix, and an absolutely thin disk ([1−3]). Let us consider the stability of these bodies when equilibrium is due to the rotation of the masses composing them.

In this case the stability of a sphere or of a rotating cylinder with respect to radial perturbations, in which no self-intersection of layers occurs, is obvious, since then each particle moves in the field of a constant mass. Under radial perturbations the angular momentum of each particle is conserved, and its stationary state corresponds to a minimum of the particle' s total energy (the Kepler problem). Consequently, the stationary state of the system of particles corresponds to a minimum of its total energy and is therefore stable.

Let us show, however, that self-intersection must necessarily arise with time. If r_0 is the Lagrangian radius of a particle and r is its true radius, then the presence of self-intersection means that $dr/dr_0 \leq 0$, i.e., particles of different layers lie at one and the same radius. Let us prescribe a perturbation of the form $r = r_0 + \varepsilon z(r_0)$, $dr/dt = 0$, where $\varepsilon z(r_0)$ is a small perturbation. Then the equations of motion determine the periodic trajectory of the particle motion

$$r = r_0 + \varepsilon z(r_0) \cos \omega(r_0) t, \qquad dr/dr_0 = 1 + \varepsilon z' \cos \omega t - \varepsilon z \omega' t \sin \omega t.$$

It is seen from this that, owing to the dependence $\omega(r_0)$, the expression for dr/dr_0 contains a term that grows with time; therefore, for any initially small perturbation, self-intersection arises. The exceptions are the homogeneous sphere and the homogeneous cylinder, for which $\omega(r_0) = \text{const}$.

The aim of the present work is to prove the stability of a sphere and a cylinder with respect to arbitrary radial perturbations (including those with self-intersection).

2. As the sphere we shall consider a system consisting of a central body of mass M and a large number of particles revolving around it in circular orbits, so that the gravitational potential of the system at any point can, with the required accuracy, be taken as spherically symmetric. Deviations from spherical symmetry are associated with the mutual influence of point masses, which can be identified with distant encounters. The latter may be neglected in the case when the time of the process is much less than the time between encounters.

The stability of such a system can be investigated by the variational method in the form ([4–7]), subsequently developed in the case of nonzero macroscopic velocity of the system in works ([8–10]), or in another form in ([11]). However, for greater clarity (without diminishing rigor), we shall construct the proof of stability in a somewhat different way.

We shall consider a sphere consisting of separate interacting spherical layers. The effective potential energy u_{eff} of the entire sys-

of the system is equal to

$$u_{\text{eff}} = \sum_i \left(-\frac{GM_i m_i}{r_i} + \frac{\mathcal{M}_i^2}{2m_i r_i^2} \right), \tag{1}$$

where M_i is the mass inside the i-th layer; m_i is the mass of the i-th spherical layer; r_i is the distance of the i-th spherical layer from the origin of coordinates; $\mathcal{M}_i^2$ is the sum of the squares of the angular momenta of the particles of the i-th spherical layer. We note that the case is not excluded when

$$M_i = \sum_i M_{ik} = 0,$$

where M_{ik} is the angular momentum of the k-th particle of the i-th layer, but

$$\mathcal{M}_i^2 = \sum_i \mathcal{M}_{ik}^2 \neq 0.$$

Since radial displacements preserve the angular momentum of each particle, $M_{ik} = \text{const}$, under these displacements $\mathcal{M}_i^2 = \text{const}$.

Let, as a result of a radial perturbation, the effective potential energy of the system become equal to u'_{eff}, which is obtained from formula (1) by replacing $r_i \to r'_i$. The stability condition of the system is

$$\Delta u_{\text{eff}} = u'_{\text{eff}} - u_{\text{eff}} > 0. \tag{2}$$

In the case when, during displacement of the layers ($r_i \to r'_i$), no intersections of them occur, the effective potential energy of the i-th layer is equal to

$$u_{\text{eff}i} = -GM_i m_i/r_i + \mathcal{M}_i^2/2m_i r_i^2. \tag{3}$$

The equilibrium condition of the i-th layer is

$$Gm_{iM}i/r_i^2 = m_i v_i^2/r_i, \tag{4}$$

which is the condition for a minimum of the effective potential energy, $\partial u_{\text{eff}i}/\partial r_i = 0$, determining the equilibrium circular orbit of radius r_i. Any displacement of the i-th spherical layer to an arbitrary point r_i' will lead to an increase in the energy (3). Since what has been stated above is valid for any layer, for the system as a whole the stability condition (2) is satisfied.

All that has been set out above is assumed to be known. We note that the stability and equilibrium of a sphere can be considered hydrodynamically, taking the pressure to be anisotropic, $P_{rr} = 0$, $P_{\theta\theta} = P_{\varphi\varphi} \neq 0$; the results obtained are the same as in the method considered here.

The kinetic treatment of the stability of a sphere is nontrivial because of the presence of poles in the perturbed distribution function. In this sense the energy method of proof presented here is distinguished by its simplicity and clarity.

In determining the equilibrium condition (4) from the equality for the minimum of the energy

$$\frac{\partial u_{\text{eff}}}{\partial r_i} = \frac{Gm_{iM}i}{r_i^2} - \frac{Gm_i}{r_i}\frac{\partial M_i}{\partial r_i} - \frac{\mathcal{M}_i^2}{m_i r_i^3} = 0$$

it is assumed that the second term on the left-hand side of the equality, $\sim \partial M_i/\partial r_i$, is equal to zero. In the case of intersection of layers, when as a result of a perturbation the value of the mass inside the i-th layer changes, the term $(Gm_i/r_i)\partial M_i/\partial r_i \neq 0$. To establish the influence of this term on the stability of the system, consider a system consisting of a central mass M and two layers intersecting as a result of perturbations:

$$u_{\text{eff}1} = -Gm_1 M/r_1 + \mathcal{M}_1^2/2m_1 r_1^2; \tag{5}$$

$$u_{\text{eff}2} = -Gm_2(M + m_1)/r_2 + \mathcal{M}_2^2/2m_2 r_2^2; \tag{6}$$

$$u'_{\text{eff}1} = -Gm_1(M + m_2)/r_1' + \mathcal{M}_1^2/2m_1 r_1'^2; \tag{7}$$

$$u'_{\text{eff}2} = -Gm_2 M/r_2' + \mathcal{M}_2^2/2m_2 r_2'^2. \tag{8}$$

Expressions (5) and (6) determine the energy of the layers before their displacements. We shall assume

$$r_1 < r_2, \qquad r_1' > r_2', \tag{9}$$

i.e., the "inner" layer after a radial perturbation becomes the "outer" one, and conversely. Determining Δu_{eff}, according to (2), we arrive at the following positive-definite form:

$$\Delta u_{\text{eff}} = \frac{1}{2} GMm_1(r_1 - r_1')^2/r_1'^2 r_1 + \frac{1}{2} G(M+m_1)m_2(r_2 - r_2')^2/r_2'^2 r_2 + Gm_1m_2(r_1' - r_2')/r_1' r_2'. \tag{10}$$

The third term in (10) determines the change in potential energy solely due to the interchange of the layers. According to (9), this term is always positive. Thus, radial perturbations that lead to the crossing of spherical layers lead to an additional increase in the effective potential energy of the system in comparison with the stretching–compression perturbations considered above.

3. The proof of stability of a cylinder infinitely extended along z with respect to radial perturbations can be carried out analogously to that considered above for the sphere. Therefore we shall restrict ourselves here to the final formula for the change in potential energy when a cylindrical layer is displaced by a distance $\Delta r = r' - r$ from the axis of rotation,

$$\Delta u_{\text{eff}} = GmM(1 - r/r')^2 > 0. \tag{11}$$

In deriving (11), the asymptotic representation of $\ln(r/r')$ as a series in a neighborhood of the point $r/r' = 1$ was used. Thus the smallness of the displacement Δr is assumed. In the case of crossing of layers, and with $M = M(r)$, a positive term appears in (11), causing additional stabilization of radial perturbations.

4. The stability of a rotating infinitely thin gravitating disk was investigated in works ([8,11]), where its instability with respect to radial and nonradial displacements was shown. Let us recall that the field intensity in the disk is determined by both the internal and the external mass.

From the foregoing one may draw the following conclusion. A necessary and sufficient condition for the stability of bodies of rotation with respect to radial perturbations is the dependence of the field intensity at any point only on the mass internal with respect to the surface $\psi(r) = \text{const}$.

In conclusion, we note that the question of the stability of a sphere and a cylinder with respect to nonradial perturbations, and the possibility of constructing a quasar model on the basis of a sphere, remains open.

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CITED LITERATURE

[1] A. M. Lyapunov, *Works on the Theory of Potential*, 1949.
[2] S. V. Kovalevskaya, *Scientific Works*, Publishing House of the Academy of Sciences of the USSR, 1948.
[3] G. Lamb, *Hydrodynamics*, Moscow-Leningrad, 1947.
[4] C. Lundquist, Phys. Rev., **83**, 307 (1951).
[5] C. Lundquist, Ark. Nat. Astr., **5**, 297 (1952).
[6] A. V. Bernstein, E. A. Frieman et al., Proc. Roy. Soc., **17**, 244 (1958); A. B. Bernshtein, E. A. Friman et al., in: *Controlled Thermonuclear Reactions*, Moscow, 1960.
[7] B. B. Kadomtsev, *Problems of Plasma Theory*, **2**, 1963.
[8] A. M. Fridman, Astr. Zhurn., **43**, 327 (1966).
[9] L. S. Solov' ev, ZhETF, **54**, 666 (1968).
[10] Ya. B. Zel' dovich, *Problems of Cosmogony*, **9**, 157 (1963).
[11] C. Hunter, Month. Not., **126**, 299 (1963).

Note: Figure translations are in progress. See original paper for figures.