ANALOGS OF STIRLING' S FORMULA

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Abstract

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MATHEMATICS

V. M. KALININ

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Stirling's formula gives an asymptotic expansion of the gamma function $\Gamma(1+x)$ as $x \to \infty$ in inverse powers of x:

$$\Gamma(1+x) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left\{\sum_{j=1}^{\nu-1} \frac{B_{j+1}}{j(j+1)x^j} + O\left(\frac{1}{x^\nu}\right)\right\},$$

$$\Gamma(1+x) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left\{1 + \sum_{j=1}^{\nu-1} \frac{b_j}{x^j} + O\left(\frac{1}{x^\nu}\right)\right\},$$

$$\frac{1}{\Gamma(1+x)} = \frac{1}{\sqrt{2\pi x}}\left(\frac{e}{x}\right)^x \left\{1 + \sum_{j=1}^{\nu-1} (-)^j \frac{b_j}{x^j} + O\left(\frac{1}{x^\nu}\right)\right\},$$

where the coefficients b_j can be found from the recurrence relations

$$b_j = \frac{1}{j}\sum_{k=0}^{j-1} \frac{b_k B_{j-k+1}}{j-k+1}, \qquad b_0 = 1.$$

Here B_{j-k+1} are Bernoulli numbers. (Somewhat different recurrence equations are indicated in (1).) The Bernoulli numbers are easily computed recursively:

$$B_j = (-)^j j \sum_{k=0}^{j-1} \frac{B_k C_{j-1}^k}{j-k+1}, \qquad B_1 = -\frac{1}{2}.$$

In applications (for example, in probability theory) it is often convenient to expand the gamma function in powers of a quantity different from x. The following theorems introduce into Stirling' s formula an arbitrary parameter θ, by the choice of which one can obtain the necessary expansions.

Theorem 1. *For $x > -1$, $\nu = 2, 3, \ldots$, and arbitrary $\theta > -x$*

$$\Gamma(1+x)=\sqrt{2\pi}\,(x+\theta)^{x+1/2}\exp\left\{-(x+\theta)+\sum_{j=1}^{\nu-1}\frac{B_{j+1}(\theta)}{j(j+1)(x+\theta)^j}+R_\nu\right\},$$

where

$$R_\nu=-\frac{1}{\nu}\int_0^1 B_\nu(t)\zeta(\nu,x+t)\,dt+\frac{1}{\nu}\int_1^\theta\frac{B_\nu(t)}{(x+t)^\nu}\,dt,$$

$$\zeta(\nu,x)=\sum_{k=1}^{\infty}\frac{1}{(x+k)^\nu},$$

$B_{j+1}(\theta)$ are Bernoulli polynomials. They may be found, for example, from the following recurrence formulas:

$$(-)^j\frac{B_j(\theta)}{j!}=\sum_{k=0}^{j-1}(-)^k\frac{B_k(\theta)}{k!}\frac{(\theta-1)^{j-k+1}-\theta^{j-k+1}}{(j-k+1)!},\qquad B_0(\theta)=1.$$

The proof of Theorem 1 is close to the proof of Lemma 1 of paper (2); the difference is that in Taylor' s formula, on which the summation theorem is based, the remainder term is taken in integral form.

Theorem 2. As $x\to\infty$, $\nu=1,2,...$, uniformly with respect to arbitrary $\theta=O(1)$, the expansions

$$\Gamma(1+x)=\sqrt{2\pi}\,(x+\theta)^{x+1/2}\exp\left\{-(x+\theta)+\sum_{j=1}^{\nu-1}\frac{B_{j+1}(\theta)}{j(j+1)(x+\theta)^j}+O\left(\frac{1}{(x+\theta)^\nu}\right)\right\},$$

$$\Gamma(1+x)=\sqrt{2\pi}\,(x+\theta)^{x+1/2}e^{-(x+\theta)}\left\{1+\sum_{j=1}^{\nu-1}\frac{b_j(\theta)}{(x+\theta)^j}+O\left(\frac{1}{(x+\theta)^\nu}\right)\right\},$$

$$\frac{1}{\Gamma(1+x)}=\frac{e^{x+\theta}}{\sqrt{2\pi}\,(x+\theta)^{x+1/2}}\left\{1+\sum_{j=1}^{\nu-1}(-1)^j\frac{b_j(1-\theta)}{(x+\theta)^j}+O\left(\frac{1}{(x+\theta)^\nu}\right)\right\},$$

hold, where $b_j(\theta)$ are polynomials of degree $2j$, satisfying the recurrence relations

$$b_j(\theta)=\frac{1}{j}\sum_{k=0}^{j-1}\frac{b_k(\theta)B_{j-k+1}(\theta)}{j-k+1},\qquad b_0(\theta)=1,$$

$$\Delta b_j(\theta) = \theta b_{j-1}(1+\theta).$$

In particular, from Theorem 2 we find, as $x \to \infty$,

$$\Gamma(1+x) = [x]!\, x^{\{x\}} \exp\left\{ \sum_{j=1}^{\nu-1} \frac{B_{j+1} - B_{j+1}(\{x\})}{j(j+1)x^j} + O\left(\frac{1}{x^\nu}\right) \right\},$$

(here $[x]$ and $\{x\}$ are the integer and fractional parts of x).

As $n \to \infty$ and $x = O(1)$, the following equality holds uniformly with respect to x:

$$\Gamma(x) = \frac{\sqrt{2\pi}\, n^{n+x+1/2} e^{-n}}{x(x+1)\cdots(x+n)} \exp\left\{ \sum_{j=1}^{\nu-1} \frac{B_{j+1}(-x)}{j(j+1)n^j} + O\left(\frac{1}{n^\nu}\right) \right\}.$$

Theorem 3. As $x \to \infty$, uniformly with respect to arbitrary $\theta = O(\sqrt{x})$ and arbitrary $a = O(1)$, the expansions

$$\Gamma(1+x) = \sqrt{2\pi}\,(x+\theta)^{x+1/2} \exp\left\{ -(x+\theta) + \frac{y^2}{2} + \sum_{j=1}^{\nu-1} \frac{v_j(y,a)}{(\sqrt{x+\theta})^j} + O\left(\frac{1}{(\sqrt{x+\theta})^\nu}\right) \right\},$$

$$\Gamma(1+x) = \sqrt{2\pi}\,(x+\theta)^{x+1/2} e^{-(x+\theta)-y^2/2} \left\{ 1 + \sum_{j=1}^{\nu-1} \frac{w_j(y,a)}{(\sqrt{x+\theta})^j} + O\left(\frac{1}{(\sqrt{x+\theta})^\nu}\right) \right\},$$

$$\frac{1}{\Gamma(1+x)} = \frac{e^{x+\theta-y^2/2}}{\sqrt{2\pi}\,(x+\theta)^{x+1/2}} \left\{ 1 + \sum_{j=1}^{\nu-1} (-i)^j \frac{w_j(iy, 1-a)}{(\sqrt{x+\theta})^j} + O\left(\frac{1}{(\sqrt{x+\theta})^\nu}\right) \right\},$$

hold, where

$$y = \frac{\theta - a}{\sqrt{x+\theta}},$$

$$v_j(y,a) = \sum_{k=0}^{1+[j/2]} \frac{B_k(a) C_{j-k+2}^k y^{j-2k+2}}{(j-k+1)(j-k+2)},$$

and $w_j(y,a)$ are polynomials of degree $3j$ in y of parity j, satisfying the recurrence relations

$$w_j(y,a) = \sum_{k=0}^{j-1} \left(1 - \frac{k}{j}\right) w_k(y,a) v_{j-k}(y,a), \qquad w_0(y,a) = 1.$$

(The imaginary unit i enters the expression $(-i)^j w_j(iy, 1-a)$ only in even powers.)

Theorems 2 and 3 are direct consequences of Theorem 1. All the assertions formulated can also be given in a complex version. Thus, Theorem 2 remains valid if x is replaced by a complex variable z as $|z| \to \infty$, uniformly with respect to an arbitrary complex quantity $\theta = O(1)$ and with respect to $|\arg z| \le \pi - \varepsilon$. Theorem 3 holds when x is replaced by z, uniformly with respect to arbitrary complex quantities $\theta = O(\sqrt[\nu]{z})$, $a = O(1)$, and with respect to $|\arg z| \le \pi - \varepsilon$. This makes it possible, in particular, to describe the asymptotic behavior of $\Gamma(1+z)$ for $z = x + iy$. For example, one may state the following result:

Theorem 4. *For $z = x \pm iy$, $y \to \infty$, the following equalities hold uniformly with respect to $x = O(1)$:*

$$\Gamma(1+z) = \sqrt{2\pi}\, y^{x+1/2} \exp\left\{ -\frac{\pi}{2} y \pm i \left[\frac{\pi}{2}\left(x + \frac{1}{2}\right) + y(\ln y - 1) \right] + \sum_{j=1}^{\nu-1} \frac{(\mp i)^j B_{j+1}(-x)}{j(j+1) y^j} + O\left(\frac{1}{y^\nu}\right) \right\},$$

$$\Gamma(1+z) = \sqrt{2\pi}\, y^{x+1/2} \exp\left\{ -\frac{\pi}{2} y \pm i \left[\frac{\pi}{2}\left(x + \frac{1}{2}\right) + y(\ln y - 1) \right] \right\} \times$$

$$\times \left\{ 1 + \sum_{j=1}^{\nu-1} (\mp i)^j \frac{b_j(-x)}{y^j} + O\left(\frac{1}{y^\nu}\right) \right\}.$$

From the complex version of Theorem 3 one can determine the asymptotic behavior of $\Gamma(1+x\pm iy)$ as $y \to \infty$ and $x = O(\sqrt[\nu]{y})$. For example, we immediately obtain

$$|\Gamma(1 + x \pm iy)| = \sqrt{2\pi}\, y^{x+1/2} e^{-\frac{\pi}{2} y} \left\{ 1 + O\left(\frac{1}{\sqrt[\nu]{y}}\right) \right\}$$

uniformly with respect to $x = O(\sqrt[\nu]{y})$.

We note that, instead of the expansions of $\Gamma(1+x)$ and $\Gamma(1+z)$, one may everywhere write, with some obvious changes, the expansions of $\Gamma(x)$ and $\Gamma(z)$.

In the last two theorems the behavior of a frequently occurring ratio is described.

Theorem 5. *As $|z| \to \infty$, uniformly with respect to an arbitrary complex $\theta = O(1)$ and with respect to $|\arg z| \le \pi - \varepsilon$, the following expansions hold:*

$$\frac{\Gamma(z+\theta)}{\Gamma(z)} = z^\theta \exp\left\{\sum_{j=1}^{\nu-1}(-)^j \frac{B_{j+1}-B_{j+1}(\theta)}{j(j+1)z^j} + O\left(\frac{1}{z^\nu}\right)\right\},$$

$$\frac{\Gamma(z+\theta)}{\Gamma(z)} = z^\theta \left\{1 + \sum_{j=1}^{\nu-1} \frac{C_\theta^{(j)}}{z^j} + O\left(\frac{1}{z^\nu}\right)\right\},$$

$$\frac{\Gamma(z)}{\Gamma(z+\theta)} = z^{-\theta} \left\{1 + \sum_{j=1}^{\nu-1}(-)^j \frac{C_{1-\theta}^{(j)}}{z^j} + O\left(\frac{1}{z^\nu}\right)\right\},$$

$$C_\theta^{(j)} = \theta(\theta-1)\cdots(\theta-j)\sum_{k=0}^{j-1} g_{jk}(\theta-j-1)\cdots(\theta-j-k),$$

$$g_{j0} = \frac{1}{j+1}, \qquad g_{j,j-1} = \frac{1}{2^j j!}, \qquad g_{jk} = \frac{(j+k)g_{j-1,k}+g_{j-1,k-1}}{j+k+1}, \quad k=1,\dots,j{-}2.$$

(The values of the polynomials $C_\theta^{(j)}$ for $\theta = j{+}1, j{+}2, \dots$ are the Stirling numbers of the first kind $C_\theta^{(j)}$, and for $\theta = -1, -2, \dots$ the Stirling numbers of the second kind $\mathfrak{S}_{-\theta}^{(j)}$.)

Theorem 6. As $|z| \to \infty$, uniformly with respect to an arbitrary complex $\theta = O(1)$ and with respect to $|\arg z| \leqslant \pi - \varepsilon$, the following expansions hold:

$$\frac{\Gamma(z+\theta\sqrt{z})}{\Gamma(z)} = z^{\theta\sqrt{z}} \exp\left\{\frac{\theta^2}{2} + \sum_{j=1}^{\nu-1}(-)^j \frac{\alpha_j}{(\sqrt{z})^j} + O\left(\frac{1}{(\sqrt{z})^\nu}\right)\right\},$$

$$\frac{\Gamma(z+\theta\sqrt{z})}{\Gamma(z)} = z^{\theta\sqrt{z}} e^{\theta^2/2} \left\{1 + \sum_{j=1}^{\nu-1}(-)^j \frac{\beta_j}{(\sqrt{z})^j} + O\left(\frac{1}{(\sqrt{z})^\nu}\right)\right\},$$

$$\frac{\Gamma(z)}{\Gamma(z+\theta\sqrt{z})} = z^{-\theta\sqrt{z}} e^{-\theta^2/2} \left\{1 + \sum_{j=1}^{\nu-1}(-)^j \frac{\gamma_j}{(\sqrt{z})^j} + O\left(\frac{1}{(\sqrt{z})^\nu}\right)\right\},$$

where

$$\alpha_j = \sum_{k=0}^{[(j+1)/2]} (-)^k \frac{C_j^k B_k \theta^{j-2k+2}}{(j-k+1)(j-k+2)},$$

$$\beta_j = \sum_{k=0}^{j-1}\left(1-\frac{k}{j}\right)\beta_k \alpha_{j-k}, \qquad \beta_0 = 1,$$

$$\gamma_j = -\sum_{k=0}^{j-1}\left(1-\frac{k}{j}\right)\gamma_k \alpha_{j-k}, \qquad \gamma_0 = 1.$$

Another analogue of Stirling' s formula is contained in work (2).

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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2. V. M. Kalinin, *Theory of Probability and Its Applications*, **12**, no. 1, 24 (1967).

Note: Figure translations are in progress. See original paper for figures.