THE STRUCTURE OF A GROUP AND THE STRUCTURE OF ITS SUBGROUPS

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Abstract

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MATHEMATICS

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THE STRUCTURE OF A GROUP AND THE STRUCTURE OF ITS SUBGROUPS

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1. All groups considered below are of finite order. We denote: $|M|$ is the cardinality of the set M; Γ_n is the set of all n-maximal subgroups in G; the set of all n-maximal solvable subgroups in G ([1]) will be denoted by the symbol Δ_n; p and q are only prime numbers; a Z_p-group is a group with cyclic Sylow p-subgroup; $\Phi(G)$ is the Frattini subgroup; $Z(G)$ is the center of the group G; H^G is the normal closure of H in G; $\lambda(m)$ is the sum of the exponents in the decomposition of m into prime factors (m is a positive integer); a group of type S is a minimal nonnilpotent group.

Definition 1. We shall call $G = PH$ a group of type S_1 if P is a noninvariant Sylow subgroup in G, while $H = Q_1 \times \cdots \times Q_n$ is the (direct) product of invariant Sylow subgroups in G, and $PQ_i/\Phi(Q_i)$ is a minimal nonabelian group. A direct product of groups of type S_1 of pairwise relatively prime orders will be called a generalized Schmidt group.

Definition 2. A direct product of Frobenius groups of pairwise relatively prime orders will be called a generalized Frobenius group.

Definition 3. We shall call G an E_n-group if, for any of its subgroups H and F, from $|H| = |F|$ and $\lambda(H) \leq n$ it follows that H and F contain the same number of subgroups. An E-group is an E_n-group for all n.

Definition 4. A D-group is a group in which every subgroup H has the following property: if A and B are subgroups in H of equal orders, then the number of subgroups lying between H and A is equal to the number of subgroups lying between H and B.

2. The results presented in this section are devoted to the influence of the generation of certain subgroups on the structure of the group.

Theorem 1. *If a nonabelian group G coincides with its commutant, then in G there exists a Z_2-subgroup H of type S and of order $2^m \cdot q$, such that $H^G = G$. In particular, the normal closure of a Sylow 2-subgroup from H in G is equal to G.*

Corollary 1. *Let a proper subgroup H be permutable with all subgroups of type S from G. Then either G is different from its commutant, or $H^G \neq G$.*

Theorem 2. *A nonsolvable group G contains such a Z_2-subgroup H of type S and of order $2^m \cdot q$, that H^G is nonsolvable, and from $HF = G$, where F is a subgroup in G, it follows that $F = G$. In particular, if all noninvariant subgroups of type S are complemented in G, then the latter is solvable.*

Theorem 3. *If, in a nonsolvable group G, any two Sylow p-subgroups generate a p-solvable subgroup, then G contains a nonsolvable subgroup of order not divisible by p (this refines one result of B. Fischer).*

3. In this section only p-groups are considered.

Definition 5. An $I(n, k)$-group is a p-group in which any two incident subgroups of indices p^n and p^{n+k} have cyclic intersection.

Lemma 1. Let every proper homomorphic image of the nonmetacyclic p-group G be metacyclic. Then G is of one of the following types: a) elementary of order p^3; b) nonabelian of order p^3 and exponent $p > 2$; c) $p = 2$ and

$$G = \mathrm{gp}\langle a, b, c \mid a^2 = b^2 = c^4 = 1,\ ab = ba,\ ac = ca,\ bc = cab\rangle.$$

Corollary 2. A 2-group is metacyclic if and only if it and all its maximal subgroups are generated by two elements.

Theorem 4. Let $G = AB$, where A and B are cyclic 2-subgroups. If G is nonmetacyclic, then its automorphism group is a 2-group and every normal divisor of it is generated by three elements.

This theorem is proved with the aid of the Itô–Ohara theorem on the product of two cyclic 2-subgroups.

We shall give a more complete description of $I(n, 0)$-groups than in ([1]), Theorem 14 (there in item (c) it was erroneously asserted that $\Phi(G)$ coincides with the commutator subgroup G'):

Theorem 5. If the $I(n, 0)$-group G has order p^m, then one of the following assertions holds: a) $m < n+3$; b) G contains a cyclic subgroup of index p; c) G is an extension of a noncyclic subgroup R of order 4 by means of the generalized quaternion group of order 2^{n+1} $(n > 1)$; G contains exactly three subgroups of orders 2 and 4, $R \subseteq \Phi(G)$, the center and the commutator subgroup are nonincident cyclic subgroups of orders 4 and 2^n, respectively; G contains exactly one noncyclic subgroup of order 8, which is abelian of type $(4, 2)$; G contains exactly two invariant cyclic subgroups of order 2^{n+1}, and the factor groups by them are cyclic; the class of G is $m - 2$.

The group c) of Theorem 5 has the property that if it has odd index in a finite group, then it always has an invariant complement.

Theorem 6. The classes of $I(n, 1)$-groups and $I(n + 1, 0)$-groups coincide.

Theorem 7. If there is exactly one maximal chain leading from the p-group G to its subgroup of order p, then either G contains a cyclic subgroup of index p, or it is a group of maximal class.

The question of whether all p-groups of maximal class satisfy the condition of the theorem remains open (for $p = 2$ this is true).

Theorem 8. Let G be a p-group of odd order. Then $\Phi(G) \subseteq Z(G)$ if and only if all subgroups of $\Phi(G)$ are invariant in G. In particular, if $\Phi(G)$ is cyclic, then $G = AB$, where A is cyclic and contains $\Phi(G)$, while B is generated by all elements of order p in G.

Corollary 3. A p-group contains a cyclic subgroup of index p if and only if it contains fewer than $p^2 + p + 1$ subgroups of index p^2. A 2-group with two generators is metacyclic if and only if it contains fewer than 11 subgroups of index 4.

Theorem 9. Let the 2-group G not be a group of maximal class. Then the number of subgroups of maximal class and of a given order contained in it is divisible by 4.

Corollary 4. Let the group G of order p^m be noncyclic and not a 2-group of maximal class. Then:

1) The number of maximal chains in G is congruent to $1 + p(m-1)$ modulo p^2.
2) The number of nontrivial subgroups of a given order contained in G is congruent to $1 + p$ modulo p^2.
3) The number of cyclic subgroups of order p^n, $n > 1$, contained in G is divisible by p.

Items 2) and 3) were previously proved for $p > 2$ by Kulakov and Miller, respectively.

Theorem 10. In a p-group the number of principal series is congruent to 1 modulo p.

In view of the fact that invariance is nontransitive, P. Hall' s enumeration principle is of little use for counting invariant subgroups; therefore we shall give its modification, more convenient for this purpose. Let R be non-

whose set of normal divisors in the p-group G, M is its cover, $|M| = p^m$. Let H_i be a subgroup of order p^i in M, and let $R(H_i)$ denote the set of all those elements R which contain H_i. Then

$$|R| = \sum |R(H_1)| - p \sum |R(H_2)| + \dots + (-1)^{i-1} p^{i(i-1)/2} \sum |R(H_i)| + \dots$$
$$\dots + (-1)^{m-1} p^{m(m-1)/2} |R(H_m)|.$$

Here the summation in the i-th term is over all H_i. Obviously, $H_m = M$.

In the theory of p-groups there is a known conjecture that an isolated subgroup (i.e., one having trivial intersection with any cyclic subgroup not contained in it) which contains an element of composite order must be maximal. The following group gives a counterexample to this conjecture:

$$G = gp\langle a, b, c \mid a^4 = b^2 = c^4 = 1,\ ab = ba,\ ac = cab,\ bc = ca^2b\rangle.$$

Then $H = gp\langle a, b\rangle$ is isolated in G, has the element a of composite order, and is not maximal in G.

4. Below we give a strengthening of a number of theorems from ([1]).

Theorem 11. If the intersection of any two nonidentity subgroups of a nonsolvable group G is 2-decomposable, then it is isomorphic to one of the following groups:

a) $PSL(2, 2^p)$, $\lambda(2^p \pm 1) < 3$; $Sz(8)$.

b) $PSL(2, 3^p)$ and $SL(2, 3^p)$, $p > 2$, $\lambda(3^p + 1) < 4$, and $\lambda(3^p - 1) < 4$ or $3^p - 1 = 2q^n$.

c) $PSL(2, p)$ or $SL(2, p)$, $p^2+1 \equiv 0(\bmod 5)$. If 4 divides $p-1$, then $\lambda(p-1) < 4$, and $p + 1 = 2q^n$ or $\lambda(p + 1) < 4$. If 4 divides $p + 1$, then $p + 1 = 2^m q$ and $q > 3$ in the case $m > 2$, or $p = 7$, and $\lambda(p - 1) < 4$ or $p - 1 = 2q_1^t$.

Theorem 12. If all elements from Δ_n are generalized Schmidt groups, then $n = 1$ and $G = G_1 \times G_2$, $(|G_1|, |G_2|) = 1$, G_2 is a generalized Schmidt group, and G_1 is isomorphic to $SL(2, 5)$ or $PSL(2, 2^p)$, where $2^p - 1$ is a Mersenne prime.

Corollary 5. If all elements from Γ_n are generalized Schmidt groups, then $n = 1$ and G is equal to G_1 from Theorem 12.

Theorem 13. Let all elements from Δ_n be generalized Frobenius groups. Then one of the following assertions holds:

a) G is a generalized Frobenius group;

b) G is isomorphic to one of the groups $PSL(2, 2^n)$ or $Sz(2^{2m+1})$;

c) $G = G_1 \times G_2$, $(|G_1|, |G_2|) = 1$, where G_2 is a group from a), and G_1 is a group from b).

Corollary 6. Let all elements from Γ_n be generalized Frobenius groups. Then G is a group from items a) or b) of Theorem 13, and in the case b) the numbers n and $2m + 1$ are prime.

Theorem 14. Let all third maximal subgroups of a nonabelian simple group G have an invariant 2-complement (or be 2-closed). Then every nonsolvable subgroup in G is semisimple (such groups, under the name of N-groups, were completely described by J. Thompson ([3])).

Theorem 15. Let all elements from Δ_3 be 2-decomposable. Then the group G is either solvable, or isomorphic to one of the groups:

a) $PSL(2, 2^p)$, $\lambda(2^p \pm 1) < 3$; $PSL(2, 3^p)$ and $SL(2, 3^p)$, $p > 2$ and $\lambda(3^p \pm 1) < 4$; $PSL(2, p)$ and $SL(2, p)$, $\lambda(p \pm 1) < 4$ or $p = 31$; $Sz(8)$; $PGL(2, p)$, $p = 5, 7$.

b) $L \times D$, where L is the icosahedral group, and D has prime order.

c) An extension of $SL(2, p)$, $p = 5, 7$, by a group of order q. Moreover, if $p = 7$, then $q = 2$ and G contains exactly one involution.

Lemma 2. A p-group which is an E_3-group is of one of the following types: 1) a cyclic group; 2) a nonabelian group of order p^3 and exponent $p > 2$; 3) a noncyclic elementary group; 4) an ordinary quaternion group.

Theorem 16. A soluble E-group has an ordered Sylow tower. Its nilpotent length does not exceed 3, and its derived length does not exceed 4 (these bounds cannot be lowered).

Theorem 17. Suppose that in a nonsoluble group G all soluble subgroups are E_3-groups. Then G contains a Hall normal divisor L such that $G/L \cong SL(2, 5)$ or $L \cong PSL(2, 2^n)$.

Lemma 3. If in a p-group G any two subgroups of the same order are contained in the same number of subgroups, then G is of one of types 1), 3), 4) of Lemma 2.

Theorem 18. For D-groups the conclusions of Theorems 16 and 17 are valid (this does not mean, however, that the classes of E-groups and D-groups coincide).

Theorem 15 substantially strengthens the result of Gagen–Janko ([2]), who found all simple groups with nilpotent third maximal subgroups. Theorem 4 gives a positive solution to a question from B. Huppert' s work on the product of cyclic 2-groups. Theorem 8 strengthens an important theorem of Ph. Hall on p-groups of odd order in which all characteristic abelian subgroups are cyclic. If d is the minimal number of generators of the group in the second part of this theorem, then G contains exactly $1 + p + \cdots + p^{d-1} + mp^d$ subgroups of order p. Here $m = 1$ if G is nonabelian of exponent p, and $m = 0$ if the exponent of G is greater than p. This substantially strengthens an old result of Taussky on the number of subgroups of order p in such a group (Taussky additionally required $p > 3$, whereas here $p > 2$).

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Note: Figure translations are in progress. See original paper for figures.