ON DUALITY IN PROBLEMS OF THE CALCULUS OF VARIATIONS

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 519.3

MATHEMATICS

A. D. IOFFE, V. M. TIKHOMIROV

ON DUALITY IN PROBLEMS OF THE CALCULUS OF VARIATIONS

(Presented by Academician A. N. Kolmogorov on 1 VIII 1967)

1. Consider the simplest problem of the calculus of variations, consisting in finding

$$S(x) = \inf I(u) = \int_T f(t,u)\,dt$$

on the set

$$U(x) = \left\{ u(t) : \int_T u(t)\,dt = x \right\}.$$

Here $T = [t_0, t_1]$, $u(t)$ is a summable real function; $f(t,u)$ is B-measurable in the aggregate of the variables. It may always be assumed ([1]) that $f(t,u)$ is convex in u. It is assumed that $|S(x)| < \infty$ for all x. Let

$$\mathcal{H}(p) = \sup_x (px - S(x)),$$

$$H(t,p) = \sup_u (pu - f(t,u))$$

be the Young transforms of the functions $S(x)$ and $f(t,u)$.

Theorem 1.

$$S(x) = \max_p \left(px - \int_T H(t,p)\,dt \right).$$

Proof. By definition,

$$S(x) + \mathcal{H}(p) \geq px \tag{1}$$

for all x and p. On the other hand, $S(x)$ is obviously convex, and, since it is defined on the whole axis, there exists a straight line $p_0 x - a$ supporting $S(x)$ at the point $\bar{x}$, i.e.,

$$p_0\bar{x} - a = S(\bar{x}); \qquad p_0 x - a \le S(x); \qquad \forall x \ne \bar{x}.$$

Consequently, $a = \mathcal{H}(p_0)$ and

$$S(\bar{x}) + \mathcal{H}(p_0) = p_0\bar{x}. \tag{2}$$

Comparing (1) and (2), we obtain

$$S(\bar{x}) = \max_p (p\bar{x} - \mathcal{H}(p)). \tag{3}$$

Finally,

$$\mathcal{H}(p) = \sup_x (px - S(x)) = \sup_x \left(px - \inf_{U(x)} \int_T f(t, u(t))\, dt \right) =$$

$$= \sup_x \sup_{U(x)} \int_T (pu(t) - f(t, u(t)))\, dt = \sup_{L_1(T)} \int_T (pu(t) - f(t, u(t)))\, dt =$$

$$= \int_T \sup_u (pu - f(t, u))\, dt = \int_T H(t, p)\, dt,$$

which, together with (3), proves the theorem.

Thus, variational problems of the indicated type are dual to problems of maximization of real functions. The theorems formulated below contain all information about the solutions of the problems under consideration.

Let

$$P = \left\{ p : \int_T H(t, p)\, dt < \infty \right\}.$$

Theorem 2. *If in the set P there exists an upper boundary point p_+, then:*

1) *there exists a point $\tau_+ \in T$, in every neighborhood of which*

$$\int_{\tau_+ - \varepsilon}^{\tau_+ + \varepsilon} H(t, p)\, dt = \infty \quad \text{for all } p > p_+;$$

2) *the functional* $I(u)$ *can be extended to the collection* $\{x(t)\}$ *of functions of bounded variation having a positive singular part concentrated at the point* τ_+, *by the formula*

$$J(x(t)) = p_+(x(\tau_+ + 0) - x(\tau_+ - 0)) + I(\dot{x}(t)).$$

An analogous theorem holds if in P there exists a lower boundary point p_-.

Theorem 3. *Let*

$$\max_p \left(px - \int_T H(t,p)\,dt \right)$$

be attained at the point p_0. *Then:*

1) *(Euler case) if* $p_0 \in \operatorname{int} P$, *then there exists* $u(t) \in U(x)$ *for which* $I(u(t)) = S(x)$, *and for almost every* t

$$p_0 u(t) - f(t, u(t)) = H(t, p_0).$$

2) *if* $p_0 = p_+$ *and* $\operatorname{int} P \neq \emptyset$, *then there exists a function of bounded variation* $x(t)$, *satisfying condition 2) of Theorem 2, such that*

$$J(x(t)) = S(x);$$

$$\int_T \dot{x}(t)\,dt \le x, \qquad x(\tau_+ + 0) - x(\tau_+ - 0) = x - \int_T \dot{x}(t)\,dt;$$

$$p_0\dot{x}(t) - f(t, \dot{x}(t)) = H(t, p_0) \quad \text{almost everywhere;}$$

3) *if* $P = \{p_0\}$, *then there exists a sequence* $x_n(t)$ *of functions of bounded variation satisfying condition 2) of Theorem 2 and such that*

$$J(x_n(t)) \to S(x);$$

$$x_n(\tau_+ + 0) - x_n(\tau_+ - 0) = \left(x - \int_T \dot{x}_n(t)\,dt \right)^+;$$

$$x_n(\tau_- + 0) - x_n(\tau_- - 0) = -\left(x - \int_T \dot{x}_n(t)\,dt \right)^-;$$

$$p_0\dot{x}_n(t) - f(t, \dot{x}_n(t)) \to H(t, p_0) \quad \text{almost everywhere.}$$

2. The formulated results may be considered within the framework of a more general theory of duality, constructed for finite-dimensional spaces by Fenchel ([2]) (see also ([3]), pp. 256–284).

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be real linear spaces. We shall say, following ([4]), that $\mathfrak{X}$ and $\mathfrak{Y}$ are in duality if there exists a bilinear functional (x, y) satisfying two conditions:

1) if $(x, y) = 0$ for all $y \in \mathfrak{Y}$, then $x = 0$;

2) if $(x, y) = 0$ for all $x \in \mathfrak{X}$, then $y = 0$.

(If $\mathfrak{X}$ is a B-space, $\mathfrak{Y} = \mathfrak{X}^*$, then $(x, y) = (x, x^*)$.)

The duality relation gives rise to a topology in each of the spaces. A basis of this topology, for example, in $\mathfrak{X}$ is formed by sets of the form

$$G(x_0, A, \varepsilon) = \{x : \ |(x, y) - (x_0, p)| < \varepsilon, \ y \in A\},$$

where A is a finite subset of $\mathfrak{Y}$. Endowed with the duality topologies, $\mathfrak{X}$ and $\mathfrak{Y}$ become locally convex linear topological spaces.

Let (S, X) be the pair formed by a real function $S(x)$, defined on a set $X \subset \mathfrak{X}$ and equal to ∞ outside X, and by the set X itself. Put

$$\mathcal{H}(y) = \sup_x((x, y) - S(x)),$$

$$Y = \{y : \ \mathcal{H}(y) < \infty\}.$$

The pair $(\mathcal{H}, Y)$ obtained in this way will be called conjugate, or dual, to (S, X), and will be denoted by $(S, X)^*$. The function $\mathcal{H}(y)$ will be called the Young transform of the function $S(x)$.

The pair (S, X) is called convex if X is convex and $S(x)$ is convex on X, and closed if $S(x)$ is lower semicontinuous. The sum of the pairs (S_1, X_1) and (S_2, X_2) will be called the pair

$$(S, X) = (S_1, X_1) + (S_2, X_2) = (S_1 + S_2, X_1 \cap X_2).$$

The pair defined by the function

$$S(x) = \inf_{x_1 + x_2 = x}(S_1(x_1) + S_2(x_2))$$

will be called the direct sum of the pairs (S_1, X_1) and (S_2, X_2):

$$(S, X) = (S_1, X_1) \oplus (S_2, X_2).$$

Theorem 4. $(S, X) = (S, X)^{**}$ if and only if (S, X) is a convex closed pair.

Theorem 5.

$$\left(\bigoplus_{i=1}^{n}(S_i, X_i)\right)^* = \sum_{i=1}^{n}(S_i, X_i)^*.$$

Under additional assumptions, the assertion dual to Theorem 5 is valid:

$$\left(\sum_{i=1}^{n}(S_i, X_i)\right)^* = \bigoplus_{i=1}^{n}(S_i, X_i)^*, \quad ({}^8, {}^9),$$

which leads to a number of meaningful consequences (for example, to modularity theorems (where Σ_{22}, S_{22} are $(p-1)\times(p-1)$-matrices)).

3. Theorem 1 may be regarded as an integral analogue of Theorem 5. We can now formulate a far-reaching generalization of Theorem 1.

Let T be a bicompact Hausdorff space with a regular measure, $\mathfrak{U}$ a separable metric space, $\mathfrak{X}$ a B-space, $\mathfrak{Y}=\mathfrak{X}^*$; $f(t,u): T\times\mathfrak{U}\to R^1$ and $\varphi(t,u): T\times\mathfrak{U}\to\mathfrak{X}$ are B-measurable, and $\mathfrak{U}(t)$ is a mapping of T into the set of subsets of $\mathfrak{U}$ with Borel graph in $T\times\mathfrak{U}$.

Let

$$U(x)=\left\{u(t)\in\mathfrak{U}(t):\ \int_T\varphi(t,u(t))\,dt=x\right\},\qquad S(x)=\inf_{U(x)}\int_T f(t,u(t))\,dt.$$

Denote by $S_1(x)$ the lower semicontinuous envelope of $S(x)$:

$$S_1(x)=\lim_{T}\inf_{V\subset\mathcal{T}}S(\xi),$$

$\mathcal{T}$ being a filter converging to x. Put

$$H(t,p)=\sup_{u(t)}((p,\varphi(t,u))-f(t,u)),\quad p\in\mathfrak{Y}.$$

Theorem 6.

$$S_1(x)=\sup_p\left((p,x)-\int_T H(t,p)\,dt\right).$$

Remark 1. If $\mathfrak{X}$ is finite-dimensional, then $S_1(x)=S(x)$ at interior points of the set X. In this case Theorems 2 and 3 can be generalized. These generalizations imply, in particular, the results of V. I. Arkin ([7]).

Remark 2. Theorem 6 is obviously generalized to the case where $\mathfrak{X}$ is a space with a cone and

$$U(x)=\left\{u(t)\in\mathfrak{U}(t):\int_T\varphi(t,u(t))\,dt\le x\right\}.$$

Its formulation (if a solution exists) in this case is analogous to the formulation of the Kuhn–Tucker theorem ([6]).

Received
23 VI 1967

CITED LITERATURE

[1] A. D. Ioffe, V. M. Tikhomirov, Tr. Moscow Math. Soc., 18, 187 (1968).
[2] W. Fenchel, *Convex Cones, Sets and Functions*, 1953.
[3] C. Karlin, *Mathematical Methods in Game Theory, Programming, and Economics*, IL, 1964.
[4] N. Bourbaki, *Topological Vector Spaces*, IL, 1959.
[5] A. Ya. Dubovitskii, A. A. Milyutin, Zhurn. vychisl. matem. i matem. fiziki, 5, No. 3, 395 (1965).
[6] K. Arrow, L. Hurwicz, H. Uzawa, *Studies in Linear and Nonlinear Programming*, IL, 1962.
[7] V. Arkin, *Cybernetics*, No. 2, 87 (1967).
[8] R. T. Rockafellar, Duke Math. J., 33, 81 (1966).
[9] J. J. Horeau, C. R., 256, 5047 (1963).

Note: Figure translations are in progress. See original paper for figures.