INVARIANT SUBSPACES OF DISSIPATIVE VOLTERRA OPERATORS

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Abstract

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INVARIANT SUBSPACES OF DISSIPATIVE VOLTERRA OPERATORS

WITH NUCLEAR IMAGINARY COMPONENTS

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In this paper we study nonunicellular, dissipative Volterra operators with nuclear imaginary components.

Introduction

A finite-dimensional operator whose matrix reduces to a single Jordan cell is characterized geometrically by the fact that the set of all its invariant subspaces are ordered by inclusion. In [1] the concept of unicellularity was generalized to infinite-dimensional operators: a Volterra operator is said to be *unicellular* if one of any two of its invariant subspaces belongs to the other. In [1-10] criteria for the unicellularity of certain classes of Volterra operators were established, and unicellular operators were also studied.

In the present paper we consider dissipative Volterra operators having a nuclear imaginary component. §1 contains known results and some auxiliary propositions which are used later. The invariant subspaces of nonunicellular operators are studied in §2 and §3. It is shown that every simple, dissipative Volterra operator with a nuclear imaginary component has a spectral function of first rank. The principal result of §3 may be interpreted as an analogue of the well-known theorem on the reduction of the matrix of a finite-dimensional operator to Jordan normal form. In the last section the unicellularity of an integral Volterra operator with a positive continuous kernel is established.

With regard to the terminology used in the paper, cf. [5] and [11-13].

§1.

1. Let A be a dissipative Volterra operator with a nuclear imaginary component acting in a separable Hilbert space $\mathfrak{H}$. There exists a Hilbert space $\mathfrak{G}$ and a completely continuous mapping K of the space $\mathfrak{G}$ into $\mathfrak{H}$ such that

$$A_I = \frac{1}{2i}(A - A^*) = KK^*. \tag{1}$$

The array $\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix}$ is called a *dissipative Volterra system* (it is henceforth referred to simply as a *system*). The system θ is called *simple* if A is a simple operator.

The operator function of a complex variable

$$W_\theta(\lambda) := E - 2iK^*(A - \lambda E)^{-1}K \tag{2}$$

is called the characteristic function of the system θ. The function $W(\lambda) = W_\theta(\lambda)$ satisfies the following conditions (cf. [5, 12]):

1°. $W(\lambda)$ is an entire function of $z = 1/\lambda$ for which the expansion

$$W(\lambda) = E + \frac{i}{\lambda} H_1 + \left(\frac{i}{\lambda}\right)^2 H_2 + \cdots \tag{3}$$

converges in norm; here the H_k $(k = 1, 2, \ldots)$ are completely continuous operators acting in the space $\mathfrak{G}$, and $\operatorname{sp} H_1 < \infty$.

$$\begin{aligned} &2°. \quad W(\lambda)W^*(\lambda) \geq E \qquad (\operatorname{Im}\lambda > 0). \\ &3°. \quad W(\lambda)W^*(\lambda) = E \qquad (\operatorname{Im}\lambda = 0). \\ &4°. \quad W(\lambda)W^*(\lambda) \leq E \qquad (\operatorname{Im}\lambda < 0). \end{aligned}$$

If a function $W(\lambda)$ satisfies the conditions 1°-3°, then it is the characteristic function of some simple system θ. The class of all operator functions satisfying the conditions 1°-3° is denoted by $\Omega_{\mathfrak{G}}^{(0)}$.

Let the function $W(\lambda) \in \Omega_{\mathfrak{G}}^{(0)}$ be expressed in the form (3). It can then be shown that H_1 is a nonnegative operator. The number $\mu = \operatorname{sp} H_1$ is called the weight of the function $W(\lambda)$. If the function $W(\lambda)$ is the characteristic function of the system

$$\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix},$$

then

$$\operatorname{sp} H_1 = 2 \operatorname{sp} KK^* = 2 \operatorname{sp} A_I. \tag{4}$$

If μ is the weight of the function $W(\lambda) \in \Omega_{\mathfrak{G}}^{(0)}$, then for all $\lambda \neq 0$ we have the estimate (cf. [2]):

$$\|W(\lambda)\| \leq e^{\frac{\mu}{|\lambda|}}. \tag{5}$$

There exists a constant $\sigma = \sigma_W$ such that for any $\epsilon > 0$ the following inequalities hold:

$$\|W(\lambda)\| < e^{\frac{\sigma+\epsilon}{|\lambda|}}, \quad |\lambda| < \delta(\epsilon), \quad \|W(\lambda_k)\| > e^{\frac{\sigma-\epsilon}{|\lambda_k|}}, \quad \lambda_k = \lambda_k(\epsilon) \to 0.$$

The number σ_W will be called the type of the function $W(\lambda)$. It follows from (5) that

$$\sigma_W \leq \mu. \tag{6}$$

Applying the theorem of S. N. Bernštein [14] to the scalar entire functions $(W(\lambda)g, g')$ $(g, g' \in \mathfrak{G},\ \lambda = 1/z)$, we obtain the estimate:

$$\|W(\lambda)\| \leq e^{\sigma_W |\operatorname{Im} \frac{1}{\lambda}|} \quad (\lambda \neq 0). \tag{7}$$

In particular, if $\sigma_W = 0$, then $W(\lambda) \equiv E$.

We have the following criterion for unicellularity (cf. [1, 5]).

Let A be a simple dissipative Volterra operator with a nuclear imaginary component; let θ be an arbitrary system containing A, and let $W(\lambda)$ be the characteristic function of the system θ. Then,

in order that the operator A be unicellular it is necessary and sufficient that $\sigma_W = 2l$, where $l = \operatorname{sp} A_I$.

Let

$$\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix}$$

be a dissipative Volterra system; let $\mathfrak{H}_0$ be an arbitrary subspace of $\mathfrak{H}$, and let P_0 be the projection operator onto $\mathfrak{H}_0$. Then the array

$$\theta_0 = \begin{pmatrix} A_0 & K_0 \\ \mathfrak{H}_0 & \mathfrak{G} \end{pmatrix},$$

where A_0 is an operator acting in $\mathfrak{H}_0$ defined by $A_0 = P_0 A$ and $K_0 = P_0 K$, is also a dissipative Volterra system which is called the *projection of the system θ onto the subspace $\mathfrak{H}_0$*; it is denoted by $\operatorname{pr}_{\mathfrak{H}_0} \theta$.

We have the following theorem (cf. [12]):

If the subspace $\mathfrak{H}_0$ is invariant under A and $\mathfrak{H}_1 = \mathfrak{H} \ominus \mathfrak{H}_0$, then

$$W_\theta(\lambda) = W_{\theta_0}(\lambda) W_{\theta_1}(\lambda) \quad (\theta_k = \operatorname{pr}_{\mathfrak{H}_k} \theta, \quad k = 0, 1). \tag{8}$$

If the system θ *is simple, then the systems* θ_k $(k = 0, 1)$ *are also simple.*

Let the function $W_1(\lambda)$ be expressed as a product

$$W_1(\lambda) = W_2(\lambda) W_3(\lambda) \quad (W_k(\lambda) \in \Omega^{(0)}_{\mathfrak{G}}, \quad k = 1, 2, 3). \tag{9}$$

The functions $W_2(\lambda)$ and $W_3(\lambda)$ are called *left* and *right divisors* of $W_1(\lambda)$, respectively, (a left divisor will simply be called a *divisor*).

If (9) holds, then it is easily seen that the following relations hold:

$$\mu_1 = \mu_2 + \mu_3, \tag{10}$$

$$\sigma_2, \ \sigma_3 \leq \sigma_1 \leq \sigma_2 + \sigma_3, \tag{11}$$

where μ_k is the weight and σ_k is the type of the function $W_k(\lambda)$ $(k = 1, 2, 3)$.

We consider the set $\mathfrak{N}$ of all divisors of the function $W(\lambda) \in \Omega^{(0)}_{\mathfrak{G}}$ and introduce a partial ordering in this set as follows: we shall say that $W_1(\lambda) \leq W_2(\lambda)$ $(W_k(\lambda) \in \mathfrak{N}, \ k = 1, 2)$ if $W_1(\lambda)$ is a divisor of $W_2(\lambda)$. The function $W(\lambda)$ is said to be *ordered* if the set $\mathfrak{N}$ is linearly ordered in the sense of the order relation just defined.

Let

$$\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix}$$

be a simple system. We consider the mapping Φ of the set $\mathfrak{M}$ of all subspaces invariant under the operator A into the set $\mathfrak{N}$ of divisors of the function $W(\lambda) = W_\theta(\lambda)$, defined by

$$\Phi(\mathfrak{H}_0) = W_{\mathrm{pr}_{\mathfrak{H}_0} \theta}(\lambda) \quad (\mathfrak{H}_0 \in \mathfrak{M}). \tag{12}$$

The following properties hold (cf. [12]):

1) Φ is the one-to-one mapping of the set $\mathfrak{M}$ onto the set $\mathfrak{N}$.

2) $\Phi(0) = E, \ \Phi(\mathfrak{H}) = W(\lambda)$.

3) $\mathfrak{H}_1 \subset \mathfrak{H}_2$ $(\mathfrak{H}_k \in \mathfrak{M}, \ k = 1, 2)$ if and only if $\Phi(\mathfrak{H}_1) \leq \Phi(\mathfrak{H}_2)$.

The following theorems follow from the above (cf. [3, 5, 12]):

If $\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix}$ is a simple system, then in order that the operator A should be unicellular it is necessary and sufficient that the function $W(\lambda) = W_\theta(\lambda)$ be ordered.

In order that the function $W(\lambda) \in \Omega^{(0)}_{\mathfrak{G}}$ be ordered, it is necessary and sufficient that its weight and type coincide.

2. Let $\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix}$ be some simple system, and let $W(\lambda) = W_\theta(\lambda)$; let $\mathfrak{M}$ be the set of subspaces invariant under the operator A, and let $\mathfrak{N}$ be the set of divisors of the function $W(\lambda)$. If $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are arbitrary subspaces in $\mathfrak{M}$, then the subspaces $\mathfrak{H}_0 = \mathfrak{H}_1 \cap \mathfrak{H}_2$ and $\mathfrak{H}_3$, which is the closure of the set of vectors of the form $f_1 + f_2$ $(f_k \in \mathfrak{H}_k,\ k = 1, 2)$, also belong to $\mathfrak{M}$. Let $W_k(\lambda) = \Phi(\mathfrak{H}_k)$ $(k = 0, 1, 2, 3)$, where Φ is the mapping of $\mathfrak{M}$ onto $\mathfrak{N}$ defined by (12). The following result follows immediately from the properties (1)-(3).

Theorem 1. If $W'(\lambda) \preccurlyeq W_k(\lambda)$ $(k = 1, 2)$, then $W'(\lambda) \preccurlyeq W_0(\lambda)$. If $W_k(\lambda) \preccurlyeq W''(\lambda)$ $(k = 1, 2)$, then $W_3(\lambda) \preccurlyeq W''(\lambda)$ $(W''(\lambda) \in \mathfrak{N})$.

By Theorem 1 the functions $W_0(\lambda)$ and $W_3(\lambda)$ are the greatest common divisor and the least common multiple of the functions $W_k(\lambda)$ $(k = 1, 2)$ respectively; they will be denoted by the symbols

$$W_0(\lambda) = (W_1, W_2), \qquad W_3(\lambda) = [W_1, W_2].$$

Theorem 2. The types σ_k and weights μ_k of the functions $W_k(\lambda)$ $(k = 0, 1, 2, 3)$ satisfy the following relations:

$$\sigma_3 = \max\{\sigma_1, \sigma_2\}, \tag{13}$$

$$\mu_0 = \mu_1 + \mu_2 - \mu_3. \tag{14}$$

We obtain the proof of (13) using (7) and (11), by repeating nearly verbatim the arguments used in establishing the sufficiency of the criterion for unicellularity in [1]. The proof of the second relation is based on the following two lemmas.

Lemma 1. Let

$$AT = TB, \tag{15}$$

where A and B are bounded linear operators with nuclear imaginary components, acting in the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{H}'$ respectively, and T is a bounded linear operator from $\mathfrak{H}'$ to $\mathfrak{H}$. If the operator $C = AT$ is completely continuous and zero is not an eigenvalue of the operators T and T^*, then

$$\operatorname{sp} A_I = \operatorname{sp} B_I. \tag{16}$$

Proof. By [5], there exists an orthonormal basis $\{\phi_\alpha\}$ in the closure $\mathfrak{H}_1$ of the range of the operator C and an orthonormal basis $\{\psi_\alpha\}$ in the closure $\mathfrak{H}_1'$ in the range of the operator C^* such that $C\psi_\alpha = s_\alpha \phi_\alpha$ and $C^*\phi_\alpha = s_\alpha \psi_\alpha$ $(s_\alpha > 0,\ \alpha = 1, 2, \ldots)$. Moreover, by (15),

$$(A\phi_\alpha, \phi_\alpha) = (B\psi_\alpha, \psi_\alpha) \qquad (\alpha = 1, 2, ...). \tag{17}$$

If $\phi \in \mathfrak{H} \ominus \mathfrak{H}_1$, then clearly $C^*\phi = 0$, and since zero is not an eigenvalue of the operator T^*, it is necessary that $A^*\phi = 0$. Therefore,

$$\operatorname{sp} A_I = \sum_{\alpha=1}^{\infty} \operatorname{Im}(A\varphi_\alpha, \varphi_\alpha). \tag{18}$$

It is proved similarly that

$$\operatorname{sp} B_I = \sum_{\alpha=1}^{\infty} \operatorname{Im}(B\psi_\alpha, \psi_\alpha). \tag{19}$$

The equality (16) follows easily from (17), (18), and (19).

Lemma 2. *Let A be a completely continuous operator with a nuclear imaginary component, acting in the Hilbert space $\mathfrak{H}$. If $\mathfrak{H}_k$ $(k = 1, 2)$ are subspaces invariant under A that satisfy the conditions: 1) $\mathfrak{H}_1 \cap \mathfrak{H}_2 = 0$, and 2) the closure of the set of vectors of the form $f_1 + f_2$ $(f_k \in \mathfrak{H}_k,\ k = 1, 2)$ coincides with $\mathfrak{H}$, then*

$$\operatorname{sp} A_I = \operatorname{sp} A_I^{(1)} + \operatorname{sp} A_I^{(2)}, \tag{20}$$

where $A^{(k)}$ is the operator induced by the operator A in $\mathfrak{H}_k$ $(k = 1, 2)$.

Proof. The equality (20) is obvious if the subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are orthogonal. In the general case we form the orthogonal sum $\mathfrak{H}' = \mathfrak{H}_1' \oplus \mathfrak{H}_2'$ of the Hilbert spaces $\mathfrak{H}_1'$ and $\mathfrak{H}_2'$ and consider the isometric mapping T_k of the subspace $\mathfrak{H}_k'$ onto $\mathfrak{H}_k$ $(k = 1, 2)$. We denote the orthogonal projection of $\mathfrak{H}'$ onto $\mathfrak{H}_k'$ by P_k $(k = 1, 2)$ and consider the operators

$$B^{(k)} = T_k^{-1} A^{(k)} T_k \quad (k = 1, 2), \tag{21}$$

$$B = B^{(1)} P_1 + B^{(2)} P_2, \tag{22}$$

$$T = T_1 P_1 + T_2 P_2, \tag{23}$$

acting in $\mathfrak{H}_k'$, $\mathfrak{H}'$, and from $\mathfrak{H}'$ to $\mathfrak{H}$, respectively. The operator T is defined on the whole space $\mathfrak{H}'$ and is bounded ($\|T\| \leq 2$). From the equation $Tf = 0$ it follows by condition (1) that $T_1 P_1 f = T_2 P_2 f = 0$, whence $P_1 f = P_2 f = 0$, i.e., $f = 0$. On the other hand, the range of T is dense in $\mathfrak{H}$, by condition (2), and hence zero is not an eigenvalue of the operator T^*.

We shall show that $TB = AT$. Indeed, from (21), (22), and (23) we have:

$$TB = (T_1P_1 + T_2P_2)(B^{(1)}P_1 + B^{(2)}P_2) = T_1B^{(1)}P_1 + T_2B^{(2)}P_2$$

$$= A^{(1)}T_1P_1 + A^{(2)}T_2P_2 = AT_1P_1 + AT_2P_2 = A(T_1P_1 + T_2P_2) = AT.$$

Since the operators $A^{(k)}$ and $B^{(k)}$ $(k = 1, 2)$ are unitarily equivalent and, as is easily seen, $A^{(k)}$ has a nuclear imaginary component, it follows that $B^{(k)}$ also has a nuclear imaginary component and $\operatorname{sp} B_I^{(k)} = \operatorname{sp} A_I^{(k)}$ $(k = 1, 2)$. Therefore, the imaginary component of the operator B is nuclear and $B_I = \operatorname{sp} B_I^{(1)} + \operatorname{sp} B_I^{(2)}$. Using Lemma 1, we find that

$$\operatorname{sp} A_I = \operatorname{sp} B_I = \operatorname{sp} B_I^{(1)} + \operatorname{sp} B_I^{(2)} = \operatorname{sp} A_I^{(1)} + \operatorname{sp} A_I^{(2)}.$$

We now proceed to the proof of the equality (14). From the obvious relations $W_0(\lambda) \leq W_k(\lambda)$ $(k = 1, 2, 3)$ it follows that

$$W_k(\lambda) = W_0(\lambda)W_k'(\lambda) \quad \left(W_k'(\lambda) \in \Omega_{\mathfrak{G}}^{(0)}, \quad k = 1, 2, 3\right). \tag{24}$$

Denoting the weight of the function $W_k'(\lambda)$ by μ_k' $(k = 1, 2, 3)$, we find from (24), on the strength of (10), that

$$\mu_k = \mu_0 + \mu_k', \quad (k = 1, 2, 3). \tag{25}$$

Further, since $W_k(\lambda) \leq W_3(\lambda)$ $(k = 1, 2)$, it follows that $W_k'(\lambda) \leq W_3'(\lambda)$. We consider the system

$$\theta' = \begin{pmatrix} A' & K' \\ \mathfrak{H}' & \mathfrak{G}' \end{pmatrix},$$

for which $W_3'(\lambda)$ is the characteristic function, and let Φ' be the mapping of the set $\mathfrak{M}'$ of all the subspaces invariant under A' onto the set $\mathfrak{N}'$ of all divisors of $W_3'(\lambda)$. It is easy to show that

$$(W_1', W_2') = E, \qquad [W_1', W_2'] = W_3'(\lambda),$$

and hence that the subspaces $\mathfrak{H}_k'$ $(k = 1, 2)$ defined by the equation $\Phi'(\mathfrak{H}_k') = W_k'(\lambda)$ satisfy the conditions of Lemma 2. Applying Lemma 2 and (4), we arrive at the equality

$$\mu_1' + \mu_2' = \mu_3'; \tag{26}$$

(14) follows from (25) and (26). This completes the proof of the theorem.

3. The following proposition, which plays an essential role in what follows, is contained in [5], although in another form.

Theorem 3. Let $W(\lambda)$ be the characteristic function of the system $\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix}$. If the inequality $\sigma < \mu$ holds, where μ is the weight and σ is the type of the function $W(\lambda)$, then for any vector $g_0 \in \mathfrak{G}$ there exists a divisor $W_0(\lambda)$ of the function $W(\lambda)$ of weight $\mu_0 = \mu - \sigma$ such that $W_0(\lambda) g_0 \equiv g_0$.

In conclusion we note that all the results of the present section that refer to left divisors can immediately be carried over to right divisors. Indeed, the functions $W(\lambda)$ and $W^*(-\bar{\lambda})$ both belong to the class $\Omega_{\mathfrak{G}}^{(0)}$ and have the same weight and type. If $W(\lambda)$ is the characteristic function of the system

$$\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix},$$

then $W^*(-\bar{\lambda})$ is the characteristic function of the system

$$\theta^* = \begin{pmatrix} -A^* & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix}.$$

§2.

Let $W(\lambda)$ be the characteristic function of the simple system $\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix}$, and let Φ be the mapping, defined in §1, of the set $\mathfrak{M}$ of the subspaces invariant under A onto the set $\mathfrak{N}$ of divisors of $W(\lambda)$.

Lemma 3. If the subspaces $\mathfrak{H}_1 \supset \mathfrak{H}_2 \supset \cdots$ are invariant under A, $\mathfrak{H}_0 = \bigcap_{j=1}^{\infty} \mathfrak{H}_j$, and $W_j(\lambda) = \Phi(\mathfrak{H}_j)$ $(j = 0, 1, \ldots)$, then

$$\lim \mu_j = \mu_0, \qquad \lim \sigma_j = \sigma_0,$$

where μ_j is the weight and σ_j the type of the function $W_j(\lambda)$ $(j = 0, 1, \ldots)$.

Proof. We use the equality $\mu_j = 2\,\mathrm{sp}(P_j K K^* P_j)$, where $KK^* = A_I$ and P_j is the orthogonal projection onto $\mathfrak{H}_j$ $(j = 0, 1, \ldots)$. The sequence of projections P_j converges to P_0 in the sense of strong convergence. Since the operator K is completely continuous, $\lim(P_{jK}) = P_0 K$ uniformly. Therefore,

$$\lim \mu_j = 2 \lim \mathrm{sp}(P_j K K^* P_j) = 2\,\mathrm{sp} \lim(P_j K K^* P_j) = 2\,\mathrm{sp}(P_0 K K^* P_0) = \mu_0.$$

For the proof of the second equality, we note first of all that, by (11), $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_0$, and hence $\lim \sigma_j \geq \sigma_0$. We consider the equalities

$$W_j(\lambda) = W_0(\lambda) W_j'(\lambda) \qquad \left(W_j'(\lambda) \in \Omega_{\mathfrak{G}}^{(0)}, \quad j = 1, 2, \dots\right)$$

and denote by μ_j' and σ_j', respectively, the weight and type of $W_j'(\lambda)$. Since, by (6), (10), and (11),

$$\sigma_j' \leq \mu_j', \qquad \mu_j' = \mu_j - \mu_0, \qquad \sigma_j \leq \sigma_0 + \sigma_j' \quad (j = 1, 2, \dots),$$

and, by what was proved above, $\lim \mu_j' = 0$, it follows that $\lim \sigma_j \leq \sigma_0$.

Lemma 4. Let the type σ of the function $W(\lambda)$ be less than its weight μ. Then for any $\epsilon > 0$ $(\epsilon < \sigma)$ there exists a divisor $W_1(\lambda)$ of the function $W(\lambda)$ of weight $\mu_1 = \sigma$ whose type σ_1 satisfies the inequality

$$\sigma_1 \geq \sigma - \epsilon.$$

Proof. There exist vectors $g_1, g_2 \in \mathfrak{G}$ such that the type of the scalar function $(W(\lambda)g_1, g_2)$ $(\lambda = 1/z)$ is not less than $\sigma - \epsilon$. Applying the result analogous to Theorem 3 for right divisors, we find a right divisor $W_2(\lambda)$ of the function $W(\lambda)$ of weight $\mu_2 = \mu - \sigma$ such that $W_2(\lambda)g_1 \equiv g_1$; we let $W(\lambda) = W_1(\lambda)W_2(\lambda)$. By (10), the weight μ_1 of the function $W_1(\lambda)$ is equal to σ. From the equality $(W(\lambda)g_1, g_2) = (W_1(\lambda)g_1, g_2)$ it follows that the type of $W_1(\lambda)$ is not less than the type of $(W(\lambda)g_1, g_2)$.

Theorem 4. If a dissipative Volterra operator A with a nuclear imaginary component is nonunicellular, then there exists a nontrivial subspace $\mathfrak{H}_0$ invariant under A in which the induced operator is unicellular.

Proof. We may assume without loss of generality that the operator A is simple. We include A in some system θ and let $W(\lambda) = W_\theta(\lambda)$. It is sufficient to show that there exists a divisor $W_0(\lambda)$ of the function $W(\lambda)$, that is distinct from E and whose type and weight coincide.

Since the operator A is nonunicellular, it follows from the basic criterion for unicellularity that $\sigma < \mu$, where μ is the weight and σ the type of $W(\lambda)$. Let ϵ be any number satisfying the inequality $0 < \epsilon < \sigma$. Using the preceding lemma, we find a divisor $W_1(\lambda)$ of the function $W(\lambda)$ of weight $\mu_1 = \sigma$, whose type σ_1 satisfies the condition $\sigma_1 \geq \sigma - \epsilon/2$. Next, we denote by $W_2(\lambda)$ a divisor of the function $W_1(\lambda)$ of weight $\mu_2 = \sigma_1$, whose type σ_2 is not less than $\sigma_1 - \epsilon/4$. Continuing in this manner, we obtain a sequence $\{W_k(\lambda)\}$ of divisors of the function $W(\lambda)$ satisfying the following conditions:

$$\begin{aligned} &1) \quad W_{k+1}(\lambda) \leq W_k(\lambda) \quad (k = 1, 2, \dots), \\ &2) \quad \mu_1 = \sigma, \ \mu_k = \sigma_{k-1} \quad (k = 2, 3, \dots), \end{aligned}$$

3) $\sigma_k \geq \sigma_{k-1} - \epsilon/2^k$ (μ_k is the weight and σ_k the type of $W_k(\lambda)$).

Let $\Phi(\mathfrak{H}_k) = W_k(\lambda)$ $(k = 1, 2, \ldots)$. By the condition (1) and the third property of the mapping Φ, the following relations hold: $\mathfrak{H}_{k+1} \subset \mathfrak{H}_k$ $(k = 1, 2, \ldots)$. If $\mathfrak{H}_0 = \bigcap_{k=1}^{\infty} \mathfrak{H}_k$, $W_0(\lambda) = \Phi(\mathfrak{H}_0)$, and μ_0 is the weight and σ_0 the type of the function $W_0(\lambda)$, then, applying Lemma 3 and the condition (2), we obtain the equality $\sigma_0 = \mu_0$. To complete the proof it remains only to note that, by the conditions (2) and (3), $\sigma_0 \geq \sigma - \epsilon$.

2. Let $W(\lambda)$ be the characteristic function of the system

$$\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix}.$$

By [5], $W(\lambda)$ can be expressed as a multiplicative integral

$$W(\lambda) = \int_0^l e^{\frac{2i}{\lambda}\, dH(t)} \qquad (H(t) = K^* E(t) K, \quad l = \operatorname{sp} A_I), \tag{27}$$

where $E(x)$ is any spectral function of the operator A that is normalized by the condition $\operatorname{sp}(K^* E(x) K) = x$ $(0 \leq x \leq l)$. As was shown in [5], there exists an operator function $P(x)$, defined almost everywhere on $[0, l]$, that satisfies the conditions:

$$1)\ P(x) \geq 0, \qquad \operatorname{sp} P(x) = 1 \qquad (x \in \mathcal{G},\ m\mathcal{G} = l); \tag{28}$$

2) for any $g, g' \in \mathfrak{G}$ the function $(P(x)g, g')$ is measurable and

$$(H(t)g, g') = \int_0^t (P(x)g, g')\, dx. \tag{29}$$

The following proposition was also established here. *If the operator A is unicellular, then the rank of $P(x)$ is equal to 1 almost everywhere on* $[0, l]$.

Lemma 5. *For each simple system*

$$\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix}$$

a spectral function $E(x)$ can be chosen so that the rank of $P(x)$ in the formulas (27)-(29) is equal to unity almost everywhere on $[0, l]$.

Proof. We consider the set $\Pi = \{\pi\}$ of all possible representations of the function $W(\lambda) = W_\theta(\lambda)$ in the multiplicative form (27), which correspond to different spectral functions of the operator A; for each representation $\pi \in \Pi$ we denote by $\delta(\pi)$ the length of the largest interval of the form $[0, \delta]$ for almost all x of which the rank of $P(x)$ is equal to one. It is clear that $0 \leq \delta(\pi) \leq l$; by Theorem 4 and the proposition given above, there exist representations π for which $\delta(\pi) > 0$. Let $P_k(x)$ be the function corresponding to the representation

$\pi_k \in \Pi$ $(k = 1, 2, \ldots)$. We shall say that π_1 precedes π_2 and write $\pi_1 < \pi_2$ if the following conditions are satisfied: 1) $\delta(\pi_1) < \delta(\pi_2)$, 2) $P_1(x) = P_2(x)$ $(x \in [0, \delta(\pi_1)])$. Under the relation $\leq$ the set Π becomes partially ordered. It is not hard to see that every linearly ordered subset of Π has an upper bound in Π. By Zorn' s lemma, Π contains a maximal element π_0. If $\delta(\pi') < l$, then it is easy to show by applying Theorem 4 that π' precedes some $\pi'' \in \Pi$. Therefore, $\delta(\pi_0) = l$; it now remains only to consider the spectral function $E_0(x)$ corresponding to the representation π_0.

The following result follows immediately from Lemma 5 (cf. [5]).

Theorem 5.* *For each simple dissipative Volterra operator A with a nuclear imaginary component there exists a spectral function $E(x)$ and a reproducing vector $h_0 \in \mathfrak{H}$ such that the linear hull of the vectors of the form $E(x)h_0$ $(0 \leq x \leq l,\ l = \operatorname{sp} A_I)$ is dense in $\mathfrak{H}$.*

§3.

1. It is known that any linear operator acting in a finite-dimensional space is either unicellular or is the direct sum of unicellular operators. We shall say that a Volterra operator A acting in a separable Hilbert space $\mathfrak{H}$ *decomposes into a direct sum of unicellular operators* $A = A_1 \dot{+} A_2 \dot{+} \cdots$ if there exists a finite or countable set of subspaces $\mathfrak{H}_k$, invariant under A, satisfying the following conditions: 1) the closure of the set of vectors of the form $f_1 + f_2 + \cdots + f_m$ $(f_k \in \mathfrak{H}_k,\ k = 1, 2, \ldots, m,\ m = 1, 2, \ldots)$ is equal to $\mathfrak{H}$; 2) the equation $f_1 + f_2 + \cdots + f_m = 0$ $(f_k \in \mathfrak{H}_k,\ k = 1, 2, \ldots, m,\ m = 1, 2, \ldots)$ implies that $f_1 = f_2 = \cdots = f_m = 0$; 3) the operator A_k induced by A in $\mathfrak{H}_k$ is unicellular $(k = 1, 2, \ldots)$.

In the present section it is shown that any simple dissipative Volterra operator with a nuclear imaginary component decomposes into a direct sum of unicellular operators.

2. Let A be a dissipative Volterra operator with a nuclear imaginary component, acting in the space $\mathfrak{H}$; let

$$\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix}$$

be some system containing A, and let $W(\lambda) = W_\theta(\lambda)$.

Lemma 6. *If the operator A is nonunicellular, then there exist nontrivial subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, invariant under A, whose intersection consists of the zero vector.*

Proof. It is sufficient to establish the existence of two divisors $W_1(\lambda)$ and $W_2(\lambda)$, of the function $W(\lambda)$, of positive weight, for which $(W_1, W_2) = E$. Using Theorem 4, we find a nontrivial subspace $\mathfrak{H}_1$ invariant under A in which the induced operator A_1 is unicellular; we denote the divisor of the function $W(\lambda)$ corresponding to $\mathfrak{H}_1$ by $W_1(\lambda)$. Since the operator A_1 is unicellular, it follows that $W_1(\lambda)$ has a unique representation in the multiplicative form.

$$W_1(\lambda) = \int_0^{l_1} e^{\frac{2i}{\lambda}\, dH_1(t)}, \tag{30}$$

where

$$(H_1(t)g, g') = \int_0^t (P_1(x)g, g')\, dx \qquad (g, g' \in \mathfrak{G}) \tag{31}$$

and by [5] the values of the function $P_1(x)$ ($x \in \mathcal{E}_1 \subset [0, l_1]$, $m\mathcal{E}_1 = l_1$) are projections onto one-dimensional subspaces in $\mathfrak{G}$. By Theorem 3, for any vector $g \in \mathfrak{G}$ there exists a divisor $W_g(\lambda)$ of the function $W(\lambda)$ of positive weight such that $W_g(\lambda)g = g$. If $W_1(\lambda)$ and $W_g(\lambda)$ have a common divisor of weight $\mu_0 > 0$, then, since the function $W_1(\lambda)$ is ordered, it follows from (30) that

$$W_0(\lambda) = (W_1, W_g) = \int_0^{l_0} e^{\frac{2i}{\lambda}\, dH_1(t)} \qquad \left(l_0 \geq \frac{1}{2}\mu_0\right) \tag{32}$$

and it is easily shown that $W_0(\lambda)g = g$. Therefore, from the expansion of $W_0(\lambda)$ in a series (cf. [15]) we obtain the equation

$$\int_0^{l_0} (P_1(x)g, g)\, dx = 0.$$

We shall show that a vector $g_0 \in \mathfrak{G}$ can be chosen so that $\int_0^l (P_1(x)g_0, g_0)\, dx > 0$ for any $l > 0$. To this end we consider some dense sequence $\{g_\alpha\}$ in $\mathfrak{G}$. By the theorem of A. Denjoy there exists a set $\mathcal{E}_0 \subset \mathcal{E}_1$ such that 1) $m\mathcal{E}_0 = l_1$ and 2) each of the functions $(P_1(x)g_\alpha, g_\alpha)$ is approximately continuous at the points of $\mathcal{E}_0$. We take an arbitrary sequence $x_k \in \mathcal{E}_0$ tending to zero, and put $P_k = P_1(x_k)$. It is clear that a vector $g_0 \in \mathfrak{G}$ can be chosen so that $P_k g_0 \neq 0$ for arbitrarily large k. If it is assumed that

$$\int_0^{l_0} (P_1(x)g_0, g_0)\, dx = 0 \qquad (l_0 > 0),$$

then

$$P_1(x)g_0 = 0 \qquad (x \in \mathcal{E}_0 \cap [0, l_0]),$$

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* In [5] this result was proved for unicellular operators.

which contradicts the inequalities $P_k g_0 \neq 0$. Putting $W_2(\lambda) = W_{g_0}(\lambda)$, we find that $(W_1, W_2) = E$.

Remark. If the imaginary component of the operator A is n-dimensional ($n < \infty$), then the imaginary component of the operator $A^{(2)}$ induced in $\mathfrak{H}_2$ ($\Phi(\mathfrak{H}_2) = W_2(\lambda)$) is at most $(n-1)$-dimensional. This follows from the relations $K_2 g_0 = 0$ and $K g_0 \neq 0$, which are easily obtained; here $K_2 K_2^* = A_I^{(2)}$ and $KK^* = A_I$.

Lemma 7. *If the operator A is nonunicellular, then there exists a subspace $\mathfrak{H}_1 \neq \mathfrak{H}$ invariant under A such that the type of the function $W_1(\lambda) = \Phi(\mathfrak{H}_1)$ is equal to the type σ of the function $W(\lambda)$.*

Proof. Since the operator A is nonunicellular, the operator $B = -A^*$ is also nonunicellular. Therefore, by Lemma 6 there exist nontrivial subspaces $\mathfrak{H}_1'$ and $\mathfrak{H}_2'$ invariant under B whose intersection is the zero vector. But then the subspaces $\mathfrak{H}_k = \mathfrak{H} \ominus \mathfrak{H}_k'$ ($k = 1, 2$) are invariant under A and the closure of the set of vectors of the form $f_1 + f_2$ ($f_k \in \mathfrak{H}_k,\ k = 1, 2$) coincides with $\mathfrak{H}$. Introducing the notation $W_k(\lambda) = \Phi(\mathfrak{H}_k)$ and applying (13), we find that the type of at least one of the functions $W_k(\lambda)$ ($k = 1, 2$) is equal to σ.

Lemma 8. *There exists a subspace $\mathfrak{H}_1$ invariant under A such that the operator $A^{(1)}$ induced in this subspace is unicellular and* $\operatorname{sp} A_I^{(1)} = \frac{1}{2}\sigma$, *where σ is the type of $W(\lambda)$.*

Proof. We may assume without loss of generality that the operator A is simple and nonunicellular. We consider the set $\mathfrak{M}'$ of all subspaces invariant under A with the property that the type of each of the functions $W'(\lambda) = \Phi(\mathfrak{H}')$ ($\mathfrak{H}' \in \mathfrak{M}'$) is equal to the type σ of the function $W(\lambda)$. The set $\mathfrak{M}'$ is partially ordered by inclusion of subspaces. Let $\mathfrak{M}_0' = \{\mathfrak{H}_\alpha\}$ be a linearly ordered subset of $\mathfrak{M}'$, and let $\mathfrak{H}_0 = \bigcap_\alpha \mathfrak{H}_\alpha$. By Lemma 3 the type of the function $W_0(\lambda) = \Phi(\mathfrak{H}_0)$ is also equal to σ, and hence $\mathfrak{H}_0 \in \mathfrak{M}'$. By Zorn' s lemma, $\mathfrak{M}'$ contains a minimal element $\mathfrak{H}_1$. Applying Lemma 7, we easily find that the operator $A^{(1)}$ induced in $\mathfrak{H}_1$ is unicellular. By (4), the proof of the lemma is now complete.

We shall say that an ordered divisor $W_1(\lambda)$ of the function $W(\lambda)$ is *maximal* if its weight μ_1 is equal to the type σ of the function $W(\lambda)$. As a corollary of Lemma 8, it follows that each function $W(\lambda) \in \Omega_{\mathfrak{G}}^{(0)}$ has at least one maximal divisor.

3. We now formulate the principal result of the present section.

Theorem 6. *Every simple dissipative Volterra operator A with a nuclear imaginary component is either unicellular or decomposes into a direct sum $A = A^{(1)} + A^{(2)} + \dots + A^{(m)}$ ($2 \leq m \leq \infty$) of unicellular operators. If the imaginary component A_I of the operator A is n-dimensional ($n < \infty$), then $m \leq n$ and the imaginary component $A_I^{(k)}$ of the operator $A^{(k)}$ is at most $(n-k+1)$-dimensional ($k = 1, 2, \dots, m$).*

Proof. Let

$$\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix}$$

be an arbitrary system containing A, and let $W(\lambda)$ be the characteristic function of the system θ. If the operator A is nonunicellular, then $\sigma < \mu$ where $\mu = 2\,\mathrm{sp}\,A_I$ is the weight and σ is the type of $W(\lambda)$. By Lemma 8, there exists a subspace $\mathfrak{H}_1$ invariant under A such that the operator $A^{(1)}$ induced in $\mathfrak{H}_1$ is unicellular and $\mu_1 = 2\,\mathrm{sp}\,A_I^{(1)} = \sigma$. Let $W_1(\lambda) = \Phi(\mathfrak{H}_1)$, where Φ is the mapping, defined by (12), of the set of subspaces invariant under A onto the set of divisors of the function $W(\lambda)$. Using Theorem 3 and literally repeating the arguments used in proving Lemma 6, we find a divisor $\widetilde{W}_2(\lambda)$ of the function $W(\lambda)$ having weight $\tilde{\mu}_2 = \mu - \sigma$ such that $(W_1, \widetilde{W}_2) = E$. We consider the subspace $\widetilde{\mathfrak{H}}_2$ invariant under A defined by $\Phi(\widetilde{\mathfrak{H}}_2) = \widetilde{W}_2(\lambda)$. Clearly $\mathfrak{H}_1 \cap \widetilde{\mathfrak{H}}_2 = 0$. Since $\mu_1 + \tilde{\mu}_2 = \mu$, it follows from (14) that $[W_1, \widetilde{W}_2] = W(\lambda)$, and hence the closure of the set of vectors of the form $f_1 + \tilde{f}_2$ ($f_1 \in \mathfrak{H}_1$, $\tilde{f}_2 \in \widetilde{\mathfrak{H}}_2$) coincides with $\mathfrak{H}$.

If the operator $\widetilde{A}_2$ induced in $\widetilde{\mathfrak{H}}_2$ is unicellular, then the operator A decomposes into the direct sum of two unicellular operators: $A = A^{(1)} + \widetilde{A}_2$. If not, we apply the same procedure to $\widetilde{A}_2$. Continuing this process of splitting off invariant subspaces $m-1$ times, we obtain subspaces $\mathfrak{H}_1, \mathfrak{H}_2, \dots, \mathfrak{H}_{m-1}, \widetilde{\mathfrak{H}}_m$ that are invariant under A and are easily seen to satisfy the following conditions: 1) the closure of the set of vectors of the form $f_1 + \dots + f_{m-1} + \tilde{f}_m$ ($f_k \in \mathfrak{H}_k$, $k = 1, 2, \dots, m-1$, $\tilde{f}_m \in \widetilde{\mathfrak{H}}_m$) is equal to $\mathfrak{H}$; 2) the equality $f_1 + \dots + f_{m-1} + \tilde{f}_m = 0$, ($f_k \in \mathfrak{H}_k$, $k = 1, 2, \dots, m-1$, $\tilde{f}_m \in \widetilde{\mathfrak{H}}_m$) implies that $f_1 = \dots = f_{m-1} = \tilde{f}_m = 0$; 3) the operator $A^{(k)}$ induced in $\mathfrak{H}_k$ ($k = 1, 2, \dots, m-1$) is unicellular; 4) $\mu_1 + \dots + \mu_{m-1} + \tilde{\mu}_m = \mu$ (μ_k and $\tilde{\mu}_m$ are the weights of $\Phi(\mathfrak{H}_k)$ and $\Phi(\widetilde{\mathfrak{H}}_m)$, respectively); 5) $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{m-1} \geq \tilde{\sigma}_m$ ($\tilde{\sigma}_m$ is the type of $\Phi(\widetilde{\mathfrak{H}}_m)$). If for some $m < \infty$ the operator $\widetilde{A}_m$ induced in $\widetilde{\mathfrak{H}}_m$ is unicellular, then the operator A decomposes into the direct sum of m unicellular operators:

$$A = A^{(1)} + \dots + A^{(m-1)} + \widetilde{A}_m.$$

Let us consider the case in which, on applying the above process, we obtain countable sets of invariant subspaces $\{\mathfrak{H}_k\}$ and $\{\widetilde{\mathfrak{H}}_k\}$ that satisfy the conditions (1)-(5) for any m. We shall show that the following decomposition holds:

$$A = A^{(1)} + \dots + A^{(k)} + \dots .$$

Since the conditions (1)-(3) are satisfied, it is sufficient to show that the closure $\mathfrak{H}_0$ of the set of vectors of the form $f_1 + f_2 + \dots + f_m$ ($f_k \in \mathfrak{H}_k$, $k = 1, 2, \dots, m$; $m = 1, 2, \dots$) coincides with $\mathfrak{H}$. We introduce the notation

$$\mathfrak{H}_0' = \mathfrak{H} \ominus \mathfrak{H}_0, \qquad \widetilde{\mathfrak{H}}_0 = \bigcap_{k=1}^{\infty} \widetilde{\mathfrak{H}}_k, \qquad W_0(\lambda) = \Phi(\mathfrak{H}_0), \qquad \widetilde{W}_0(\lambda) = \Phi(\widetilde{\mathfrak{H}}_0),$$

and consider the equation $W(\lambda) = W_0(\lambda)W_0'(\lambda)$. If μ_0, μ_0', and $\tilde{\mu}_0$ are the weights of the functions $W_0(\lambda)$, $W_0'(\lambda)$, and $\widetilde{W}_0(\lambda)$, respectively, then by (10)

$$\mu_0 + \mu_0' = \mu. \tag{33}$$

Using Lemma 2, we can show that

$$\mu_0 = \sum_{k=1}^{\infty} \mu_k, \tag{34}$$

and, by Lemma 3,

$$\tilde{\mu}_0 = \lim \tilde{\mu}_k. \tag{35}$$

Comparing the equalities (33)-(35) and the condition (4), we find that $\mu_0' = \tilde{\mu}_0$. Since $\lim \mu_k = 0$, it follows by the condition (5) that $\tilde{\sigma}_0 = \lim \tilde{\sigma}_k = 0$. Since, by Lemma 3, $\tilde{\sigma}_0$ is the type of the function $\widetilde{W}_0(\lambda)$, we arrive at the equation $\tilde{\mu}_0 = 0$, from which it follows that $\widetilde{W}_0(\lambda) = E$, and hence $\mathfrak{H}_0' = 0$.

To complete the proof in the finite-dimensional case, it remains only to use the remark following Lemma 6.

§4.

1. Let the function $W(\lambda)$ belong to the class $\Omega_{\mathfrak{G}}^{(0)}$. We denote by μ the weight and by σ the type of the function $W(\lambda)$. It is obvious that the weight of any ordered divisor of the function $W(\lambda)$ does not exceed σ. The existence of a maximal divisor for each function $W(\lambda)$ in $\Omega_{\mathfrak{G}}^{(0)}$, i.e., of an ordered divisor whose weight is equal to the type of $W(\lambda)$, was established in the preceding section. It can be shown that if the function $W(\lambda)$ is not ordered, then it has an infinite set of distinct maximal divisors (cf. [10, 16]).

We assume that $\mu/2 < \sigma < \mu$ and consider two maximal divisors $W_1(\lambda)$ and $W_1'(\lambda)$ of the function $W(\lambda)$. It follows from Theorem 2 that $W_1(\lambda)$ and $W_1'(\lambda)$ have a common divisor $W_0(\lambda)$ of weight $\mu_0 = 2\sigma - \mu$ (we also use the fact (cf. [2]) that every function of positive weight μ_1 has a divisor of any weight $0 < \mu_0 < \mu_1$. It is easily seen that the function $W_0(\lambda)$ does not depend on the choice of the maximal divisors.

We consider the equation $W(\lambda) = W_0(\lambda)\widetilde{W}_0(\lambda)$ and denote by $\tilde{\mu}_0$ and $\tilde{\sigma}_0$, respectively, the weight and type of $\widetilde{W}_0(\lambda)$. By (10), $\tilde{\mu}_0 = 2(\mu - \sigma)$. We shall show that $\tilde{\sigma}_0 = \mu - \sigma = \tilde{\mu}_0/2$. Applying the arguments used in proving Lemma 6, we find a divisor $W_2(\lambda)$ of the function $W(\lambda)$ of weight $\mu_2 = \mu - \sigma$ such that

$$(W_1, W_2) = E,$$

where $W_1(\lambda)$ is any maximal divisor of $W(\lambda)$. Let

$$\theta = \begin{pmatrix} A & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix}$$

be some simple system of which $W(\lambda)$ is the characteristic function, and let Φ be the mapping of the set of subspaces invariant under A onto the set of divisors of $W(\lambda)$. If $\mathfrak{H}_k$ $(k = 0, 1, 2)$ are invariant subspaces under A defined by $\Phi(\mathfrak{H}_k) = W_k(\lambda)$ and $\mathfrak{H}_3$ is the smallest subspace containing $\mathfrak{H}_0$ and $\mathfrak{H}_2$, then the subspaces $\widetilde{\mathfrak{H}}_k = \mathfrak{H} \ominus \mathfrak{H}_k$ $(k = 0, 1, 2, 3)$ are invariant under the operator $-A^*$. It is easy to verify that $\widetilde{\mathfrak{H}}_1 \cap \widetilde{\mathfrak{H}}_3 = 0$ and that $\widetilde{\mathfrak{H}}_0$ is the smallest subspace containing $\widetilde{\mathfrak{H}}_1$ and $\widetilde{\mathfrak{H}}_3$. If $\tilde{\mu}_k$ $(k = 1, 3)$ is the weight of the function $\widetilde{W}_k(\lambda) = W_k^{-1}(\lambda)W(\lambda)$, then, by (10), $\tilde{\mu}_k = \mu - \mu_k$; since $\mu_3 = \mu_0 + \mu_2 = \sigma = \mu_1$, it follows that $\tilde{\mu}_3 = \tilde{\mu}_1$. Taking account of the fact that $W^*(-\lambda)$ is the characteristic function of the system

$$\theta^* = \begin{pmatrix} -A^* & K \\ \mathfrak{H} & \mathfrak{G} \end{pmatrix},$$

and applying (13), we find that $\tilde{\sigma}_0 \leq \tilde{\mu}_1 = \mu - \sigma$. On the other hand, the equation $W(\lambda) = W_0(\lambda)\widetilde{W}_0(\lambda)$ implies that $\sigma \leq \sigma_0 + \tilde{\sigma}_0$ $(\sigma_0 = \mu_0 = 2\sigma - \mu)$, whence $\tilde{\sigma}_0 \geq \mu - \sigma$.

The following result has thus been proved.

Theorem 7. *If the inequality $\mu/2 < \sigma < \mu$ is satisfied, where μ is the weight and σ the type of the function $W(\lambda) \in \Omega_{\mathfrak{G}}^{(0)}$, then there exists an ordered function $W_0(\lambda) \in \Omega_{\mathfrak{G}}^{(0)}$ of weight $\mu_0 = 2\sigma - \mu$ and a function $\widetilde{W}_0(\lambda) \in \Omega_{\mathfrak{G}}^{(0)}$ of weight $\tilde{\mu}_0$ and type $\tilde{\sigma}_0$, related by $\tilde{\sigma}_0 = \tilde{\mu}_0/2 = \mu - \sigma$, such that*

$$W(\lambda) = W_0(\lambda)\widetilde{W}_0(\lambda). \tag{36}$$

Moreover, the function $W_0(\lambda)$ is uniquely determined and is a common divisor of all maximal divisors of the function $W(\lambda)$.

The following result is proved similarly.

Theorem 8. *Under the hypotheses of Theorem 7 there exists an ordered function $W^{(0)}(\lambda) \in \Omega_{\mathfrak{G}}^{(0)}$ of weight $\mu^{(0)} = 2\sigma - \mu$ and a function $\widetilde{W}^{(0)}(\lambda) \in \Omega_{\mathfrak{G}}^{(0)}$ of weight $\tilde{\mu}^{(0)}$ and type $\tilde{\sigma}^{(0)}$, related by $\tilde{\sigma}^{(0)} = \tilde{\mu}^{(0)}/2 = \mu - \sigma$, such that*

$$W(\lambda) = \widetilde{W}^{(0)}(\lambda)W^{(0)}(\lambda). \tag{37}$$

The function $W^{(0)}(\lambda)$ is uniquely determined and is a common right divisor of all maximal right divisors of the function $W(\lambda)$.

2. Let the function $W(\lambda) \in \Omega^{(0)}_{\mathfrak{G}}$ be factored as follows:

$$W(\lambda) = W_1(\lambda) W_2(\lambda) \quad (W_k(\lambda) \in \Omega^{(0)}_{\mathfrak{G}}, \quad k = 1, 2).$$

The $W_k(\lambda)$ $(k = 1, 2)$ are ordered since $W(\lambda)$ is. The converse is in general not true (cf. [10]). We express $W(\lambda)$ in multiplicative form:

$$W(\lambda) = \int_0^{l} e^{\frac{2i}{\lambda}\, dH(t)}, \tag{38}$$

where $H(t)$ is an operator function with values in $\mathfrak{G}$ satisfying the following conditions: $H(0) = 0$, $H(t') \leq H(t'')$ $(0 \leq t' < t'' \leq l)$, and $\operatorname{sp} H(t) = t$ $(0 \leq t \leq l)$; we put

$$W_1(\lambda) = \int_0^{x_0} e^{\frac{2i}{\lambda}\, dH(t)}, \qquad W_2(\lambda) = \int_{x_0}^{l} e^{\frac{2i}{\lambda}\, dH(t)} \quad (0 < x_0 < l). \tag{39}$$

If we denote the type of the function $W(x_1, x_2, \lambda) = \int_{x_1}^{x_2} e^{\frac{2i}{\lambda}\, dH(t)}$ $(0 \leq x_1 < x_2 \leq l)$ by $\sigma(x_1, x_2)$, then

$$\sigma(x_1, x_2) \leq 2(x_2 - x_1).$$

Lemma 9. *Let the functions* $W_k(\lambda)$ $(k = 1, 2)$ *be ordered. If the function* $W(\lambda) = W_1(\lambda) W_2(\lambda)$ *is not ordered, then*

$$\sigma(x_0 - x, x_0 + x) = 2x \left(0 < x \leq \delta_0 = \frac{\mu - \sigma}{2}\right), \tag{40}$$

where $\mu = 2l$ *is the weight and* $\sigma = \sigma(0, l)$ *is the type of* $W(\lambda)$.

Proof. By the conditions of the lemma, $\mu_k \leq \sigma < \mu$ $(k = 1, 2)$, where $\mu_1 = 2x_0$ is the weight of

$W_1(\lambda)$ and $\mu_2 = \mu - 2x_0$ is the weight of $W_2(\lambda)$. Since $\mu_1 + \mu_2 = \mu$,

$$\mu_k \geq \mu - \sigma \qquad (k = 1, 2). \tag{41}$$

We express $W(\lambda)$ in the form (36). Noting that $\mu_1 + \sigma - \mu \leq 2\sigma - \mu$, we find, by Theorems 2 and 7, that the functions $W_1(\lambda)$ and $W_0(\lambda)$ have a common divisor of weight $2l_0 = \mu_1 + \sigma - \mu$; cancelling this on both sides of (38) and taking into account (39), we find that

$$W'(\lambda) = \int_{l_0}^{x_0} e^{\frac{2i}{\lambda}}\, dH(t) \int_{x_0}^{l} e^{\frac{2i}{\lambda}}\, dH(t) = W_1'(\lambda) W_2(\lambda), \tag{42}$$

$$\mu' = \mu - 2l_0, \qquad \mu_1' = \mu_1 - 2l_0, \tag{43}$$

where μ' and μ_1' are the weights of the functions $W'(\lambda)$ and $W_1'(\lambda)$, respectively. Using (36), (38), and (42), we find for the type σ' of the function $W'(\lambda)$ that

$$\sigma' \leq \mu_0 - 2l_0 + \tilde{\sigma}_0 = \sigma - 2l_0, \qquad \sigma \leq 2l_0 + \sigma',$$

and hence

$$\sigma' = \sigma - 2l_0 = \mu - \mu_1 = \mu_2. \tag{44}$$

Applying Theorem 8 to the function $W'(\lambda)$, we write it in the form

$$W'(\lambda) = W_1'(\lambda) W_2(\lambda) = W_0'(\lambda) \widetilde{W}_0'(\lambda), \tag{45}$$

where $W_0'(\lambda)$ is an ordered function of weight $\mu_0' = 2\sigma' - \mu'$ and $\widetilde{W}_0'(\lambda)$ is a function of the class $\Omega_{\tilde{\sigma}}^{(0)}$ whose weight $\tilde{\mu}_0'$ and type $\tilde{\sigma}_0'$ are related by

$$\tilde{\sigma}_0' = \frac{1}{2}\tilde{\mu}_0' = \mu' - \sigma'. \tag{46}$$

Repeating the above arguments for the case of right divisors, we can cancel both sides of (42) and (45) by the common right divisor of the functions $W_2(\lambda)$ and $\widetilde{W}_0'(\lambda)$ of weight $2l_0' = \mu_2 + \sigma' - \mu'$. Noting that by (44)

$$\mu_0' = 2\sigma' - \mu' = \mu_2 + \sigma' - \mu' = 2l_0',$$

we arrive at the equation

$$\int_{l_0}^{x_0} e^{\frac{2i}{\lambda}} \, dH(t) \int_{x_0}^{x_0 - l_0'} e^{\frac{2i}{\lambda}} \, dH(t) = \int_{x_0 - \delta_0}^{x_0 + \delta_0} e^{\frac{2i}{\lambda}} \, dH(t) = \widetilde{W}_0'(\lambda),$$

from which it follows that

$$\sigma(x_0 - \delta_0, \ x_0 + \delta_0) = 2\delta_0.$$

This completes the proof of (40) for $x = \delta_0$.

On considering the non-ordered function $W(x_0 - x, \ x_0 + \delta_0, \lambda)$ $(0 < x < \delta_0)$ with weight $2(x + \delta_0)$ and type $-2\delta_0$ and applying the result that has already been proved, we easily obtain the inequality

$$\sigma(x_0 - x, x_0 + x) = 2x \qquad (0 < x < \delta_0).$$

Lemma 10. *Let the function*

$$W(\lambda) = \int_0^l e^{\frac{2i}{\lambda}t}\, dH(t)$$

be expressed in the form (36). If the type $\sigma(x)$ of the function

$$W(x, \lambda) = \int_0^x e^{\frac{2i}{\lambda}t}\, dH(t)$$

satisfies the inequality $\sigma(x) > x$ $(0 < x \leq l)$, then for every divisor $W_{01}(\lambda)$ of the function $W_0(\lambda)$ there exists an $x_1 \in [0, l]$ such that

$$\int_0^{x_1} e^{\frac{2i}{\lambda}t}\, dH(t) = W_{01}(\lambda)\widetilde{W}_{01}(\lambda),$$

where $\widetilde{W}_{01}(\lambda)$ is some function in $\Omega^{(\delta)}$, of weight $\tilde{\mu}_{01} \geq 0$ and type $\tilde{\sigma}_{01} = \tilde{\mu}_{01}/2$.

Proof. Using Theorem 7, we express $W(x, \lambda)$ in the form

$$W(x, \lambda) = W_0(x, \lambda)\widetilde{W}_0(x, \lambda), \tag{47}$$

where $W_0(x, \lambda)$ is an ordered function of weight $\mu_0(x) = \sigma_0(x) = 2(\sigma(x) - x)$, and $\widetilde{W}_0(x, \lambda)$ is some function in the class $\Omega^{(\delta)}$, whose weight $\tilde{\mu}_0(x)$ and type $\tilde{\sigma}_0(x)$ are related by $\tilde{\sigma}_0(x) = \tilde{\mu}_0(x)/2$ (if $\sigma(x) = 2x$, then $\widetilde{W}_0(x, \lambda) = E$).

We denote by $\nu(x)$ $(0 \leq x \leq l)$ the weight of the function $(W_0(\lambda), W_0(x, \lambda))$. Obviously $\nu(0) = 0$ and $\nu(l) = \mu_0$. We shall show that $\nu(x)$ is continuous on the segment $[0, l]$. To this end we consider arbitrary $x_1, x_2 \in [0, l]$ $(x_1 < x_2)$ and let ν_0 be the weight of the function $(W_0(x_1, \lambda), W_0(x_2, \lambda))$. Since, by Theorem 7, $W_0(x_k, \lambda)$ is a divisor of the maximal divisor of the function $W(x_k, \lambda)$ $(k = 1, 2)$ and $W(x_1, \lambda)$ is a divisor of $W(x_2, \lambda)$, we easily obtain the following estimate from Theorem 2:

$$\nu_0 \geq \min\{\sigma(x_1) + \sigma(x_2) - 2x_2, \mu_0(x_1), \mu_0(x_2)\}. \tag{48}$$

We note that if $\nu(x_1) < \nu_0$ (or $\nu(x_2) < \nu_0$), then, since the functions $W_0(\lambda)$ and $W_0(x_k, \lambda)$ $(k = 1, 2)$ are ordered, it is necessary that $\nu(x_1) = \nu(x_2)$. Therefore, in the case that $\nu(x_1) \neq \nu(x_2)$,

$$\nu_0 \leq \nu(x_k) \leq \mu_0(x_k) \qquad (k = 1, 2).$$

Thus, in any case

$$|\nu(x_1) - \nu(x_2)| \le \max\{\mu_0(x_1), \mu_0(x_2)\} - \nu_0. \tag{49}$$

Since

$$0 \le \sigma(x_2) - \sigma(x_1) \le 2(x_2 - x_1), \qquad 0 \le \nu_0 \le \mu_0(x_k) \qquad (k = 1, 2),$$

it follows by (48) that

$$0 \le \mu_0(x_k) - \nu_0 \le 2(x_2 - x_1) \qquad (k = 1, 2). \tag{50}$$

From (49) and (50) it follows that $\nu(x)$ is continuous on $[0, l]$.

Let $W_{01}(\lambda)$ be a divisor of the function $W_0(\lambda)$ of the weight μ_{01} $(0 < \mu_{01} < \mu_0)$. Then there exists a

number x_1 $(0 < x_1 < l)$ such that $\nu(x_1) = \mu_{01}$ and $\nu(x) \ne \mu_{01}$ for $x > x_1$. From the definition of $\nu(x)$ and the fact that the function $W_0(\lambda)$ is ordered, it follows that

$$(W_0(\lambda), W_0(x_1, \lambda)) = W_{01}(\lambda).$$

Let us assume that $\mu_0(x_1) > \mu_{01} = \nu(x_1)$. Using (50), we take an $x_2 > x_1$, where the difference $x_2 - x_1$ is sufficiently small, such that the inequality $\nu_0 > \nu(x_1)$ is satisfied. Then, by the above remark,

$$\nu(x_2) = \nu(x_1) = \mu_{01},$$

which contradicts the choice of x_1. Hence,

$$W_0(x_1, \lambda) = W_{01}(\lambda)$$

and the assertion of the lemma follows from (47) if we put $\widetilde{W}_{01}(\lambda) = \widetilde{W}_0(x_1, \lambda)$.

The following result is proved similarly.

Lemma 11. Let the function

$$W(\lambda) = \int_0^l e^{\frac{2i}{\lambda}\, dH(t)}$$

be expressed in the form (37). If the type $\sigma(x)$ of the function

$$W(x, \lambda) = \int_x^l e^{\frac{2i}{\lambda}\, dH(t)}$$

satisfies the inequality $\sigma(x) > l - x$ $(0 \leq x < l)$, then for every right divisor $W_{02}(\lambda)$ of the function $W^{(0)}(\lambda)$ there exists an $x_2 \in [0, l]$ such that

$$\int_{x_2}^l e^{\frac{2i}{\lambda}\, dH(t)} = \widetilde{W}_{02}(\lambda) W_{02}(\lambda),$$

where $\widetilde{W}_{02}(\lambda)$ is some function in $\Omega_\theta^{(0)}$ of weight $\tilde{\mu}_{02} \geq 0$ and type $\tilde{\sigma}_{02} = \tilde{\mu}_{02}/2$.

Lemma 12. Let the function

$$W(\lambda) = \int_0^l e^{\frac{2i}{\lambda}\, dH(t)}$$

be expressed in the form of a product $W(\lambda) = W_1(\lambda) \dots W_n(\lambda)$, where

$$W_k(\lambda) = \int_{t_{k-1}}^{t_k} e^{\frac{2i}{\lambda}\, dH(t)} \qquad (0 = t_0 < t_1 < \dots < t_n = l).$$

If, for any $0 \leq x_1 < x_2 \leq l$, the type $\sigma(x_1, x_2)$ of the function

$$W(x_1, x_2, \lambda) = \int_{x_1}^{x_2} e^{\frac{2i}{\lambda}\, dH(t)}$$

satisfies the inequality

$$\sigma(x_1, x_2) > x_2 - x_1,$$

then

$$\sigma = \sum_{k=1}^n \sigma_k,$$

where σ is the type of $W(\lambda)$ and σ_k is the type of $W_k(\lambda)$ $(k = 1, 2, \dots, n)$.

Proof. It is sufficient to prove the result for $n = 2$. Assume the contrary, i.e., that $\sigma < \sigma_1 + \mu_2$. Applying Theorems 7 and 8, we consider the representations

$$W_1(\lambda) = \int_0^{t_1} e^{\frac{2i}{\lambda}}\, dH(t) = \widetilde{W}_0^{(1)}(\lambda) W_0^{(1)}(\lambda),$$

$$W_2(\lambda) = \int_{t_1}^{l} e^{\frac{2i}{\lambda}}\, dH(t) = W_0^{(2)}(\lambda)\widetilde{W}_0^{(2)}(\lambda),$$

where the function $W_0^{(k)}(\lambda)$ $(k = 1, 2)$ is ordered and the type of the function $\widetilde{W}_0^{(k)}(\lambda)$ $(k = 1, 2)$ is equal to half its weight. We denote by $W_1'(\lambda)$ and $W_2'(\lambda)$, the maximal right divisor of $W_1(\lambda)$ and the maximal left divisor of $W_2(\lambda)$, respectively. It follows easily from our assumptions that the type σ' of the product $W'(\lambda) = W_1'(\lambda)W_2'(\lambda)$ is less than its weight $\mu' = \sigma_1 + \sigma_2$, and hence the function $W'(\lambda)$ is not ordered. Taking account of the fact that $W_0^{(1)}(\lambda)$ is a right divisor of $W_1'(\lambda)$ and $W_0^{(2)}(\lambda)$ is a left divisor of $W_2'(\lambda)$, and using Lemma 9 and the basic criterion for ordering, we can easily show that the function $W_0^{(1)}(\lambda)W_0^{(2)}(\lambda)$ is also not ordered. Therefore, using Lemma 9 again, we can find a right divisor $W_{01}(\lambda)$ of the function $W_0^{(1)}(\lambda)$ of positive weight and a left divisor $W_{02}(\lambda)$ of the function $W_0^{(2)}(\lambda)$ of the same weight such that the type of the function $W_{01}(\lambda)W_{02}(\lambda)$ is equal to half its weight. By Lemmas 10 and 11, there exist numbers x_1 and x_2 $(0 \le x_1 < t_1 < x_2 \le l)$ such that

$$\int_{x_1}^{t_1} e^{\frac{2i}{\lambda}}\, dH(t) = \widetilde{W}_{01}(\lambda)W_{01}(\lambda), \qquad \int_{t_1}^{x_2} e^{\frac{2i}{\lambda}}\, dH(t) = W_{02}(\lambda)\widetilde{W}_{02}(\lambda),$$

where the type of each of the functions $\widetilde{W}_{01}(\lambda)$ and $\widetilde{W}_{02}(\lambda)$ is equal to half the weight of this function. The equality

$$W(x_1, x_2, \lambda) = \widetilde{W}_{01}(\lambda)W_{01}(\lambda)W_{02}(\lambda)\widetilde{W}_{02}(\lambda)$$

implies that $\sigma(x_1, x_2) \le x_2 - x_1$. This, however, contradicts the hypotheses of the lemma.

3. We consider the integral operator

$$Af(x) = 2i \int_{x}^{l} f(t)K(t, x)\, dt \tag{51}$$

with symmetric positive kernel $K(t, x)$ in the Hilbert space $L^{(2)}(0, l)$. Using the results of part 2 of the present section, we shall establish a sufficient condition for the operator (51) to be unicellular; this condition generalizes earlier known conditions (cf. [2, 6, 7, 10]).

Theorem 9. *If the function $K(t, x)$ is continuous in the region $0 \le x \le l$, $0 \le t \le l$ and, for all $x \in [0, l]$, the function $K(x, x)$ is different from zero, then the operator (51) is unicellular.*

Proof. We may assume without loss of generality that $K(x,x) \equiv 1$. By the conditions imposed above, A is a simple dissipative Volterra operator with a nuclear imaginary component, and $\operatorname{sp} A_I = l$. Using Mercer' s theorem, we express the kernel $K(t,x)$ in the form of a uniformly convergent series

$$K(t,x) = \sum_{i=1}^{\infty} \varphi_i(t)\overline{\varphi_i(x)} \qquad (0 \le x \le l,\ 0 \le t \le l), \tag{52}$$

where the functions $\varphi_i(x)$ $(i = 1, 2, \ldots)$ are continuous on $[0, l]$. We consider the matrix operator $P(x) = \|\overline{\varphi_i(x)}\varphi_j(x)\|_{i,j=1}^{\infty}$, acting in the Hilbert space $l^{(2)}$. Since

$$\sum_{i=1}^{\infty} |\varphi_i(x)|^2 \equiv 1, \tag{53}$$

it follows that $P(x)$ is an orthogonal projection onto a one-dimensional subspace in $l^{(2)}$. There exists a system θ containing A (cf. [2]) such that

$$W(\lambda) = \int_0^l e^{\frac{2i}{\lambda}}\, dH(t) \qquad \left(H(t) = \int_0^t P(x)\, dx \right) \tag{54}$$

is the characteristic function for θ. Therefore, to prove the theorem it is sufficient to show that the function $W(\lambda)$ is ordered.

Using the continuity of the functions $\varphi_i(x)$ $(i = 1, 2, \ldots)$ and the uniform convergence of the series (53), we can easily show that the function $P(x)$ is uniformly continuous on $[0, l]$ in the sense of the operator matrix norm. We denote by $\sigma(x_1, x_2)$ the type of the function

$$W(x_1, x_2, \lambda) = \int_{x_1}^{x_2} e^{\frac{2i}{\lambda}}\, dH(t).$$

Since the series expansion for the scalar function $\xi W(x_1, x_2, \lambda)\xi^*$ $(\xi \in l^{(2)},\ \|\xi\| = 1)$ is of the form

$$\xi W(x_1, x_2, \lambda)\xi^* = 1 + \frac{2i}{\lambda} \int_{x_1}^{x_2} \xi P(x)\xi^*\, dx + \cdots,$$

it follows, if we apply a theorem of S. N. Bernštein [14], that

$$\sigma(x_1, x_2) \ge 2 \int_{x_1}^{x_2} \xi P(x)\xi^*\, dx \qquad (\xi \in l^{(2)},\ \|\xi\| = 1,\ 0 \le x_1 < x_2 \le l). \tag{55}$$

By the uniform continuity of $P(x)$, for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon)$ such that

$$\|P(x') - P(x'')\| < \varepsilon \qquad (|x' - x''| < \delta(\varepsilon), \quad x', x'' \in [0, l]); \tag{56}$$

(55) and (56) easily imply the estimate

$$\sigma(x_1, x_2) \geq 2(1 - \varepsilon)(x_2 - x_1) \qquad (0 < x_2 - x_1 < \delta(\varepsilon)). \tag{57}$$

In particular, we can find a number $\delta_0 > 0$ such that

$$\sigma(x_1, x_2) > x_2 - x_1 \qquad (0 < x_2 - x_1 < \delta_0). \tag{58}$$

We shall show that the function $W(a, b, \lambda)$ $(0 \leq a < b \leq l)$ is ordered if $b - a < \delta_0$. To this end we partition the segment $[a, b]$ by points $a = t_0 < t_1 < \cdots < t_n = b$ in such a manner that $\max(t_k - t_{k-1}) < \delta(\epsilon)$, and consider the equation

$$W(a, b, \lambda) = \prod_{k=1}^{n} W(t_{k-1}, t_k, \lambda).$$

Using (57), (58), and Lemma 12, we arrive at the inequality

$$\sigma(a, b) = \sum_{k=1}^{n} \sigma(t_{k-1}, t_k) \geq 2(1 - \epsilon)(b - a),$$

whence, since $\epsilon > 0$ is arbitrary and the obvious inequality $\sigma(a, b) \leq 2(b - a)$ holds, it follows that

$$\sigma(a, b) = 2(b - a).$$

To complete the proof, it is sufficient to again apply Lemma 12, after decomposing $W(\lambda)$ into the product

$$W(\lambda) = \prod_{k=1}^{m} W(x_{k-1}, x_k, \lambda) \quad (0 = x_0 < x_1 < \cdots < x_m = l, \ \max(x_k - x_{k-1}) < \delta_0)$$

of ordered functions.

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Note: Figure translations are in progress. See original paper for figures.

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