ON MINIMA OF THE NORM FUNCTION IN IMAGINARY QUADRATIC FIELDS

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Abstract

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ON MINIMA OF THE NORM FUNCTION IN IMAGINARY QUADRATIC FIELDS

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Let $k = Q(\sqrt{-\Delta})$; $-\Delta$ be the discriminant of k, $\Delta > 0$; $h(\Delta)$ the number of ideal classes of k; $\mathfrak{a}$ an integral ideal of k; K_ν $(\nu = 1, \dots, h)$ the ideal classes in k; $a_\nu = \min N\mathfrak{a}$ for $\mathfrak{a} \in K_\nu$; $a(\Delta) = \max a_\nu$ $(\nu = 1, \dots, h)$. As is known, a_ν is the first coefficient of the reduced form

$$Q_\nu(x, y) = a_\nu x^2 + b_\nu xy + c_\nu y^2,$$

where $\Delta = 4a_\nu c_\nu - b_\nu^2$ and $N\mathfrak{a} = Q_\nu(x, y)$ for $\mathfrak{a} \in K_\nu$. Yu. V. Linnik showed (^{1}), on the basis of ergodic considerations, that

$$A(x) = \sum_{a_\nu \le x} 1 = \frac{6}{\pi^2} x L(1, \chi_1)(1 + o(1)), \tag{1}$$

where $\chi_1(n) = (-\Delta/n)$, $x = \lambda\sqrt{\Delta}$, $\Delta \to \infty$, and $\Delta^{-1/4+\varepsilon} \le \lambda \le 3^{-1/2}$.

Therefore there can exist only a finite number of values of Δ such that

$$a(\Delta) \le \lambda\sqrt{\Delta},$$

if $\lambda < 3^{-1/2}$. However, it has not yet been possible to find an effective bound for such values of Δ, since relation (1) is proved with the aid of Siegel' s ineffective theorem on the real zero of $L(s, \chi_1)$. A recently published result of Baker (^{2}), however, makes it possible to obtain effective results concerning the behavior of the quantity $a(\Delta)$.

In this note the following is proved.

Theorem. For any fixed value $h = h(\Delta)$ there exist constants $\varkappa$ and Δ_0, depending on h, such that for $\Delta \ge \Delta_0$ the function

$$a(\Delta) \ge (\lg \Delta)^{-\varkappa}\sqrt{\Delta}.$$

The constants $\varkappa$ and Δ_0 are effectively computable.

Proof. Let $\left(\frac{D}{\Delta}\right) = 1$, $D > 0$, $\chi(n)$ be an even character mod$D$; $X(n) = \chi(n)\left(\frac{-\Delta}{n}\right)$. Then, arguing analogously to the way this is done in (^{3}), we obtain for $\sigma > 1$

$$L(2s, \chi^2) = \frac{1}{2}\sum_{\nu=1}^{h}\sideset{}{'}\sum_{(x,y)} \frac{\chi(Q_\nu(x,y))}{Q_\nu^s(x,y)} =$$

$$= \zeta(2s)\prod_{p|D}\left(1 - \frac{1}{p^{2s}}\right)\sum_{\nu=1}^{h}\chi(a_\nu)a_\nu^{-s} + \sum_{\nu=1}^{h} a_\nu^{-s}\Phi_\nu\left(\frac{b_\nu}{2a_\nu}\right), \tag{2}$$

where

$$\Phi_\nu(z) = \sum_{y=1}^{\infty}\sum_{x=-\infty}^{+\infty}\frac{\chi(Q_\nu(x,y))}{\left((x+zy)^2 + \Delta(4a_\nu)^{-1}y^2\right)^s}.$$

The function $\Phi_\nu(z)$ is periodic; its period is equal to D, and therefore it can be represented by a Fourier series

$$\Phi_\nu(z) = \sum_{r=-\infty}^{+\infty} A_r(s,\nu)\exp\frac{2\pi i}{D}rz,$$

where

$$A_r(s,\nu) = D^{-1}\int_0^D \Phi_\nu(t)\exp\frac{2\pi i}{D}rt\,dt \quad (r = 0, \pm 1, \pm 2, ...).$$

Direct calculations show that

$$A_0(s,\nu) = D^{-1}\left(\frac{\Delta}{4}\right)^{1/2-s} I(s)\sum_{\nu=1}^{h} a_\nu^{s-1}\sum_{t=1}^{D}\sigma_\nu(t)\zeta\left(2s-1, \frac{t}{D}\right), \tag{3}$$

where

$$I(s) = \int_{-\infty}^{+\infty}(1+v^2)^{-s}\,dv, \qquad \sigma_\nu(t) = \sum_{l=0}^{D-1}\chi(Q_\nu(l,t)).$$

Using the identity, as $s \to 1+0$,

$$\zeta\left(2s-1, \frac{t}{D}\right) = (2(s-1))^{-1} - \gamma + \frac{\lg D}{D} - \sum_{m=1}^{D-1}\rho^{mt}\lg(1-\rho^{-m}) + O(1),$$

where γ is Euler’ s constant, ρ is a primitive root of the equation $x^D = 1$, we can pass to the limit $s \to 1$ in the identities (2) and (3):

$$L(1,\chi)L(1,X)=\pi^2\alpha\sum_{\nu=1}^{h}\frac{\chi(a_\nu)}{a_\nu}+D^2\pi\Delta^{-1/2}\sum_{\nu=1}^{h}\lg a_\nu\sum_{t=1}^{D}\sigma_\nu(t)$$
$$-2\pi D^{-2}\Delta^{-1/2}\sum_{\mu=1}^{D-1}\lg(1-\rho^{-\mu})\sum_{\nu=1}^{h}\sum_{t=1}^{D}\sigma_\nu(t)\rho^{t\mu}+O\left(\Delta^{-1/2}\exp\left(\frac{-\pi\sqrt{\Delta}}{a(\Delta)D}\right)\right), \tag{4}$$

where α is an algebraic number.

We shall now show that, by choosing χ, we can obtain the equalities

$$\sum_{t=1}^{D}\sigma_\nu(t)=0 \quad (\nu=1,\dots,h). \tag{5}$$

If Δ is odd, we put $D=8$, and $\chi(n)=(8/n)$; a direct count justifies (5). If $\Delta\equiv 8 \text{ or } 12 \pmod{16}$, then among the first $[\lg h/\lg 2]+2$ odd primes we choose q, which is prime to Δ; this can be done by the well-known theorem of Gauss, since $2^{i-1}\le h$, where i is the number of distinct prime divisors of Δ. Let χ be an odd complex character $\mathrm{mod} q$; then X will be an odd complex character $\mathrm{mod} q\Delta$. For such characters there exist formulas (5), allowing us to transform the left-hand side of (4), and, using a remark of Chowla (6), we shall also satisfy (5). After simple transformations (4) becomes the inequality

$$\sum_{\nu=1}^{(q-1)/2}a_\nu\lg\sin\frac{\pi\nu}{q}+\beta\pi\ll\Delta^{-1/2}\exp\frac{-\pi\sqrt{\Delta}}{a(\Delta)q}, \tag{6}$$

where β, a_ν are algebraic numbers whose heights are $\ll\Delta^{c_1}$, with c_1 depending only on h; moreover, for sufficiently large Δ, the number $\beta\neq 0$ (Lemma XV from (4)), and since π is multiplicatively independent of $\sin\pi\nu/q$ $(\nu=1,\dots,(q-1)/2)$, all the hypotheses of Baker’ s theorem (2) are satisfied for the left-hand side of (6), while the right-hand side contradicts (2) for sufficiently large Δ, if one puts $\chi=q+2+\varepsilon$.

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REFERENCES

1. Yu. V. Linnik, *Ergodic properties of algebraic fields*, L., 1967, pp. 126–127.

2. A. Baker, *Mathematatika*, 14, Part I, No. 27 (1967).
3. H. Stark, *Michigan Math. J.*, 14, No. 1 (1967).
4. H. Heilbronn, *Quart. J. Math.*, Oxf. Ser., 5, No. 18 (1934).
5. T. Hasse, *Lectures on number theory*, Moscow, 1953, p. 422.
6. S. Chowla, *Det Kong. Norske vid. selsk. fornnand.*, 40, No. 12 (1967).

Note: Figure translations are in progress. See original paper for figures.