SOME EXTREMAL PROBLEMS

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Abstract

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SOME EXTREMAL PROBLEMS

FOR UNIVALENT ANALYTIC FUNCTIONS

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In this note we consider extremal problems for certain classes of analytic functions, connected with problems in the theory of quasiconformal mappings.

§ 1. Let, in the class S of univalent analytic functions in the disk $U: \ |z| < 1$,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

the problem be considered of finding the extremum of the real functional

$$I(f) = I[f(\zeta), \overline{f(\zeta)}, f'(\zeta), \overline{f'(\zeta)}, \ldots, f^{(m)}(\zeta), \overline{f^{(m)}(\zeta)}] \tag{1}$$

for fixed $\zeta \in U$, where I is a continuously differentiable function of the variables u_l, v_l, $f^{(l)}(\zeta) = u_l + iv_l$, $l = 0, 1, \ldots, m$, and

$$\sum_{l=0}^{m} |\partial I / \partial f^{(l)}| > 0.$$

By applying the variational method of M. Schiffer–G. M. Goluzin ([3,7]), it is established that every function $w = f_0(z)$ extremal for $I(f)$ maps the disk U onto a domain $B_0 = f_0(U)$ without exterior points, which is the w-plane cut along a finite number of analytic arcs (of which at least one goes to ∞). Denote by Σ the set of all extremal functions. We shall consider here the case when the boundary of the domain B_0 consists of one arc γ with endpoints $b \neq \infty$ and ∞. Then the function $f_0(z)$ is analytic everywhere on the circle $|z| = 1$, except at the point $z_0 = f_0^{-1}(\infty)$, at which $f_0(z)$ has an algebraic singularity. By virtue of univalence, in a neighborhood of the point $z = z_0$ the function $f_0(z)$ has an expansion of the form

$$f_0(z) = (z - z_0)^{-2} \sum_{s=0}^{\infty} d_s (z - z_0)^s, \qquad d_0 \neq 0.$$

Consider the function $\omega = g_0(z) = \sqrt{f_0(z) - b}$, $b = f_0(z_1)$, where z_1 is the point of the circle $|z| = 1$ corresponding to the endpoint b of the boundary slit, at which $f_0'(z)$ has a simple zero, and a fixed branch of the root is taken. Using the smoothness of the function $g_0(z)$ on the circle $|z| = 1$, it is not hard to construct its quasiconformal continuation into the domain $U_1 : \ |z| > 1$, i.e., a quasiconformal automorphism $\omega = G(z)$ of the extended z-plane, which for $|z| > 1$ satisfies the Beltrami equation $\omega_{\bar{z}} = \mu(z)\omega_z$, $\|\mu\|_{L_\infty(U_1)} \equiv k(f) < 1$, and for $|z| \leq 1$ coincides with $g_0(z)$. Returning to the function $f_0(z)$, we obtain a quasiconformal homeomorphism of the z-plane onto the two-sheeted covering of the w-plane, glued along γ, coinciding with $f_0(z)$ for $|z| \leq 1$.

Introduce the class $O(g_0)$ of quasiconformal automorphisms $\omega = f(z)$ of the extended complex z-plane satisfying the conditions: 1) the mappings f are conformal in the disk $|z| < 1$; 2) $k(f) \leq q < 1$; 3) $f(0) = g_0(0)$, $f'(0) = g_0'(0)$, $f^{(p)}(\zeta) = g_0^{(p)}(\zeta)$, $p = 0, 1, \dots, m$; 4) $f(z_1) = 0$, $f(z_0) = \infty$.

For some $q < 1$ the class $Q(g_0)$ is nonempty. Consider the problem of finding $\inf k(f) = k_0$ in the class $Q(g_0)$.

We shall also consider the problem of finding $\inf k(f) = k_\beta$ in the class $Q_\beta(g_0)$ of quasiconformal homeomorphisms $\omega = f(z)$ of the extended complex z-plane, slit along a fixed arc $\beta : \ \theta' < \theta < \theta''$ (where $\arg z_0 < \theta' < \theta'' < \arg z_1$) of the circle $|z| = 1$, satisfying conditions 1)–4).

Lemma 1. *The class $Q_\beta(g_0)$ is compact.*

With the aid of appropriate variations of quasiconformal mappings ([5,6]) it is established that the mapping $\omega = f_\beta(z)$, extremal in the class $Q_\beta(g_0)$ (i.e., such that $k(f_\beta) = k_\beta$), maps the z-plane with a slit along the arc β onto a domain D_β having no exterior points, and moreover the plane measure of the boundary of the domain D_β is zero.

Theorem 1. *If, for some nondegenerate arc β, the function $f_1(z) = f_\beta^2(z) + b$ belongs to the set Σ for $|z| < 1$, then $k_\beta = k_0 = 0$, and the function $\omega = f_1(z)$ maps the disk $|z| < 1$ onto the ω-plane with a rectilinear slit γ_1.*

The proof of this theorem, which relates the extremal functions of the problems under consideration, rests on the following lemma, analogous to the corresponding results of ([5,6]).

Lemma 2. *Let the quasiconformal mapping $\omega = f_\beta(z)$ be extremal in the class $Q_\beta(g_0)$. Then either f_β is an analytic function, or (for $k_\beta > 0$) there exists a rational function*

$$\psi(\omega)=\sum_{s=0}^{m}\frac{A_s}{\omega(\omega-a)(\omega-\omega_0)^{s+1}}+\sum_{j=1}^{2}\frac{B_j}{\omega(\omega-a)^{j+1}}\not\equiv 0, \tag{2}$$

where $A_s=\text{const}$, $B_j=\text{const}$, $\omega_0=g_0(\xi)$, $a=g_0(0)$, *such that the characteristic* $\mu_\beta(\omega)$ *of the inverse mapping* $z=f_\beta^{-1}(\omega)$ *at the points* $\omega=f_\beta(U_1)$ *is equal to* $\mu_\beta(\omega)=k_\beta\psi(\omega)/|\psi(\omega)|$.

The function $\psi(\omega)$ *is determined up to a constant positive factor.*

It is proved that, under the hypotheses of Theorem 1, the second case in the assertion of Lemma 2 cannot occur and $k_\beta=k_0=0$. For this one uses the fact that the boundary of the domain D_β contains (in the set-theoretic sense) an analytic arc Γ_β corresponding to the arc β of the circle $|z|=1$. Applying the special variation of the ω-plane constructed in ([2]), No. 3, we obtain that, under the assumption $k_\beta>0$, on the arc Γ_β the inequality $\psi(\omega)d\omega^2>0$ must hold, and this contradicts (2).

§ 2. Let in the complex z-plane there be singled out $n\geq 1$ distinct finitely connected domains $D_1,D_2,\dots,D_n$, bounded by nondegenerate Jordan curves, and suppose that $\overline{D}_i\cap\overline{D}_j=\emptyset$ for $i\neq j$. Put $D=\bigcup_{j=1}^{n}D_j$. Denote by $Q(D,q)$ the class of quasiconformal homeomorphisms $w=f(z)$, $f(\infty)=\infty$, of the z-plane, conformal in the domains $D_1,\dots,D_n$, with dilatation $K(f)\leq q<\infty$ ($K(f)=[1+k(f)]/[1-k(f)]$), which carry prescribed finite points $z^1,z^2,\dots,z^m$, $m\geq 2$, not lying in D, into prescribed finite points $w^1,w^2,\dots,w^m$, respectively. Under the assumption that the class $Q(D,q)$ is nonempty, consider the following problem.

Problem $\widetilde{A}$. Let $z_1,z_2,\dots,z_l$ be fixed points lying in D, and let $\alpha_1,\alpha_2,\dots,\alpha_l$ be nonnegative integers. In the class $Q(D,q)$, find the maximum of the real functional

$$F(f)=F(w_{0,1},w_{1,1},\dots,w_{\alpha_1,1};\,w_{0,2},w_{1,2},\dots,w_{\alpha_2,2};\,\dots;\,w_{0,l},w_{1,l},\dots,w_{\alpha_l,l}), \tag{3}$$

where $w_{p,j}=f^{(p)}(z_j)=u_{p,j}+iv_{p,j}$, $p=0,1,\dots,\alpha_j$; $j=1,2,\dots,l$, and

the function F is continuously differentiable with respect to $u_{p,j}$ and $v_{p,j}$, and

$$\sum_{j=1}^{l}\sum_{p=0}^{\alpha_j}|\partial F/\partial w_{p,j}|>0.$$

We shall assume that $q>1$ (for $q=1$ the class $O(D,q)$ consists of a single linear mapping). Introduce the notation $\{\varphi(\xi)\}_{r,j}=\varphi^{(p)}(z_j)$.

Theorem 2. *A quasiconformal homeomorphism* $w=f_0(z)$, *for which the maximum of* $F(f)$ *in the class* $Q(D,q)$ *is attained, has the following properties:*

there exists a rational function $\psi_0(w)$, which may have simple poles and, moreover, only at the points $w^1, w^2, \dots, w^m$, and a constant χ, $0 \le \chi \le 2\pi$, such that the mapping $z = f_0^{-1}(w)$, inverse to the extremal one, has characteristic $\mu_0(w) = (f_0^{-1})_{\overline{w}}/(f_0^{-1})_w$, which is equal to zero for $w \in f_0(D)$, and at the points $w \notin f_0(D)$ has the form

$$|\mu_0(w)| = (q-1)/(q+1), \qquad \arg\mu_0(w) = -\arg\left[e^{i\chi}\psi^*(w) + \psi_0(w)\right], \tag{4}$$

where

$$\psi^*(w) = \sum_{j=1}^{l}\sum_{p=0}^{\alpha_j} F_{w_{p,j}}\{(f_0(\xi) - w)^{-1}\}_{p,j}. \tag{5}$$

In particular, if $F(f) = F(w_{0,1}, \dots, w_{0,l})$, then

$$\psi^*(w) = \sum_{j=1}^{l} F_{w_{0,j}}(w_{0,j} - w)^{-1}.$$

The proof of this theorem is carried out by the method of paper ([4]), using results analogous to Lemma 2.

Problem A can also be considered (analogously) for quasiconformal mappings of a torus on which global coordinates, varying in the complex plane, have been introduced. In this case the function F must be invariant with respect to $w_{0,1}, w_{0,2}, \dots, w_{0,l}$.

When concrete functionals and domains are considered, various distortion theorems are obtained for mappings of the classes under consideration. For example, for the class $Q(0,1;U;1+\varepsilon)$ of quasiconformal automorphisms $w = f(z)$ of the complex z-plane with normalization $f(0) = 0$, $f(1) = 1$, $f(\infty) = \infty$, conformal in the disk $U : |z| < 1$, with characteristics $\mu(z) = f_{\overline{z}}/f_z$, $\|\mu\|_{L_\infty(|z|>1)} \le \varepsilon$, for small ε the following is valid.

Theorem 3. *If $f(z) = \sum_{k=1}^{\infty} a_k z^k$ is the restriction to $|z| < 1$ of a mapping $w = f(z) \in Q(0,1;U;1+\varepsilon)$, then, up to quantities of order ε^2, the estimates*

$$1 - M\varepsilon \le |a_1| \le 1 + M\varepsilon, \qquad M = 4\pi^{-1}\int_0^1 K(r)\,dr \simeq 2.3; \tag{6}$$

$$|a_n| \le 2\varepsilon/(n-1), \qquad n = 2, 3, \dots,$$

hold, where $K(r)$ is the complete elliptic integral of the first kind. The constants M and $2/(n-1)$ cannot be improved.

To obtain the estimates (6), the known variational formula is used

$$f(z) = z - \frac{z(z-1)}{\pi} \iint_{|\zeta|>1} \frac{\mu(\zeta)\, d\xi\, d\eta}{\zeta(\zeta-1)(\zeta-z)} + O(\varepsilon^2), \qquad |z| \le R < \infty, \qquad \zeta = \xi + i\eta, \tag{7}$$

where the estimate of the remainder term depends on R. The form of the extremal mappings for which equalities are attained in (6) is established with the aid of a theorem of P. P. Belinskii (see ([1]), Theorem 1).

In general, for small $\varepsilon = q - 1$, Theorem 2 makes it possible to obtain, for a broad class of functionals, estimates that are sharp in order with respect to ε. Indeed, if $F(\varphi) = 0$ for $\varphi(z) = z$, then in the class under consideration $\max F(f) = O(\varepsilon)$.

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[1] P. P. Belinskii, *Siberian Math. J.*, **1**, No. 3, 303 (1960).
[2] P. A. Biluta, *DAN*, **171**, No. 1, 9 (1966).
[3] G. M. Goluzin, *Mat. Sb.*, **19** (61), No. 2, 203 (1946).
[4] S. L. Krushkal, *DAN*, **171**, No. 4, 784 (1966).
[5] S. L. Krushkal, *Siberian Math. J.*, **8**, No. 2, 313 (1967).
[6] S. L. Krushkal, *DAN*, **179**, No. 5 (1968).
[7] M. Schiffer, *Am. J. Math.*, **65**, No. 2, 341 (1943).

Note: Figure translations are in progress. See original paper for figures.