ON THE STABILITY OF INVERSE PROBLEMS OF GEOMETRICAL SEISMOLOGY

GEOPHYSICS

1968

SovietRxiv

Abstract

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UDC 550.3

GEOPHYSICS

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ON THE STABILITY OF INVERSE PROBLEMS OF GEOMETRICAL SEISMOLOGY

(Presented by Academician M. A. Sadovskii on 16 III 1967)

I. **1.** The inverse problem of geometrical seismology consists in determining the velocity of propagation of a seismic impulse in the Earth from its hodograph. We assume that the Earth is spherically symmetric and that the velocity depends only on the distance to the center. Depending on where the source is located —on the surface or at depth—two different problems arise. We shall formulate each of them mathematically and recall the information on uniqueness (see (1)) and on the conditions for existence of a solution (see (2)) that are necessary for understanding the theorems on stability of solutions given below.

2. Case of a surface source. The problem consists in solving, with respect to $u(y)$, the system of equations

$$X(p) = \int_0^{Y(p)} \frac{pu(y)\,dy}{\sqrt{1-p^2u^2(y)}}, \qquad T(p) = \int_0^{Y(p)} \frac{dy}{u(y)\sqrt{1-p^2u^2(y)}}, \qquad p \in (0,1), \tag{1}$$

where $Y(p) = \inf\{y,\, pu(y) \geq 1\}$.

The solution is sought in the class of positive, piecewise twice smooth $u(y)$, bounded on any segment of the half-axis $y \in [0,\infty)$ and unbounded on the whole half-axis. In addition, it is assumed that $u(0) = 1$ and that $u(y)$ forms only a finite number of waveguides $j_i = (y_i, \bar{y}_i)$ (for the precise definition see (1)).

If $u(y)$ forms n waveguides, then the function $\tau(p) = T(p) - pX(p)$* is continuous everywhere except at the values $p_1 > p_2 > \cdots > p_n$, where it has jumps $\sigma_i = \tau(p_i - 0) - \tau(p_i + 0)$, $1 \leq i \leq n$.

The following alternative holds:

1) if the functions $X(p)$ and $T(p)$ satisfy the conditions of Theorem II.4 of (2)**, then there exists a set $u^\nu(y)$ of solutions of system (1), whose graphs fill the figure G (see Fig. 1);

* More precisely, $\tau(p) = T(p) - pX(p)$ on a set dense in $(0,1)$ where $T(p)$ and $X(p)$ are finite. At the remaining points $\tau(p)$ is extended by continuity, except for the values p_i, $i = 1, 2, \dots, n$, where it is not defined.

** The conditions of this theorem are:

A. The functions $X(p)$ and $T(p)$: 1) are positive; 2) are differentiable almost everywhere; 3) $T'(p) - pX'(0) = 0$ almost everywhere on $(0,1)$; 4) for all p where $T(p)$ and $X(p)$ are not differentiable (except, perhaps, for a finite number of them), $X(p \pm 0) = X(p) = T(p \pm 0) = T(p) = \infty$.

B. The function $\tau(p) = T(p) - pX(p)$: 1) decreases monotonically; 2) $\tau(1-0) = 0$; 3) is continuous everywhere except the points p_i, $p_1 > p_2 > \dots > p_n$, where it has jumps $\sigma_i = \tau(p_i - 0) - \tau(p_i + 0)$.

C. The function

$$\Phi(q) = \frac{2}{\pi} \int_q^1 \frac{X(p)\,dp}{\sqrt{p^2 - q^2}}:$$

1) is finite for all $q \in (0,1)$; 2) is increasing; 3) $\Phi(+0) = \infty$; 4) there exists $c > 0$ such that everywhere on (p_{k+1}, p_k), where $\Phi'(q)$ is finite, the inequality

$$\Phi'(q) < -\frac{cq}{\sqrt{p_k^2 - q^2}}, \qquad 1 < k \le n, \quad p_{n+1} = 0;$$

holds; 5) the function $g(y)$, inverse to $\Phi(q)$, is piecewise twice smooth.

D. The function

$$\tau(p) + \int_p^1 \sqrt{z^2 - p^2}\, d\Phi(z)$$

is continuously differentiable for $p \ne p_i$, $i = 1, 2, \dots, n$.

2) if the conditions of Theorem II.4 from $(^2)$ are not satisfied, then no solution of the system (1) exists.

Put $V(p) = \sup\limits_\nu Y_\nu(p)$ and $N(p) = \inf\limits_\nu Y_\nu(p)$, where $Y_\nu(p) = \inf\{y,\ pu_\nu(y) \ge 1\}$. We shall determine whether $V(p)$ and $N(p)$ are determined stably from $X(p)$ and $T(p)$.

3. **The case of a deep source.** The problem reduces to solving, with respect to $H(r)$, the system of equations

$$X_1(p) = \int_0^{\mathcal{P}^{-1}} \frac{pr\,dH(r)}{\sqrt{1 - p^2 r^2}}, \qquad T_1(p) = \int_0^{\mathcal{P}^{-1}} \frac{dH(r)}{r\sqrt{1 - p^2 r^2}}, \qquad p \in [0, \mathcal{P}). \quad (2)$$

The solution is sought in the class of nondecreasing functions; it is assumed that $H(0) = 0$.

Let us note that $H(r)$ is related to $u(y)$ as follows: $H(r) = \mathrm{mes}\{y,\ y \in [0,d],\ u(y) \le r\}$, where the number d depends on the depth of the source. Therefore $u(y)$,

Fig. 1
Fig. 1

Fig. 2
Fig. 2

$y \in [0,d]$, is determined from $X_1(p)$ and $T_1(p)$, generally speaking, nonuniquely. In (^{1}) it is shown that $H(r)$, on the contrary, is found from any one of the functions $X_1(p)$ and $T_1(p)$ in a unique way.

The conditions that are necessary and sufficient for the existence of a solution of the system (2) are given by Theorem III.4 in (^{2})*. We shall prove here that $H(r)$ is determined stably from $T_1(p)$.

The sets $\Gamma\{2X(p), 2T(p)\},\ p \in (0,1)$, and $\Gamma_1\{X_1(p), T_1(p)\},\ p \in [0,\mathcal{P}]$, are curves on the x,t plane. We agree to call them, respectively, hodographs of a surface and a deep source, if the pairs of functions $X(p), T(p)$ and $X_1(p), T_1(p)$ satisfy the conditions of Theorems II.4 and III.4 (^{2}), respectively.

II. Stability in the Case of a Surface Source

1. Theorem 1. Let a hodograph Γ_0 and a sequence of hodographs $\Gamma_i,\ i = 1,2,\ldots$, be given. Let $X_i(p)$ converge to $X_0(p)$ in $L[q_0,1]$, where $0 < q_0 < 1$. Then the corresponding sequence $N(p)$ will converge to $N_0(p)$ in $L[q_0,1]$.

Proof. From (1) we have

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* It is required that: a) for any m the quadratic forms

$$\sum_0^m b_{i+j} x_i x_j, \qquad \sum_0^m (3b_{i+j+1} - 2b_{i+j} - b_{i+j+2}) x_i x_j;$$

b) the functions $X_1(p), T_1(p)$ be differentiable for $p \in [0,\mathcal{P}]$; c) $T_1'(p) - pX_1'(p) = 0$, d) $X_1(0) = 0$. The numbers $b_i,\ i = 1,2,\ldots$, are determined from the system

$$\beta_{2k+1} = \frac{k!}{(2k+1)!} \sum_{i=1}^{k+1} \frac{(2k-i+1)!}{(k-i+1)!}\, b_i, \qquad k = 0,1,2,\ldots,$$

where

$$\beta_i = \int_0^1 v^i T_1(v\mathcal{P})\, dv, \qquad i = 1,2,\ldots; \qquad b_0 = T_1(0)/2.$$

$$N_j(q) \equiv \Phi_j(q) = \frac{2}{\pi}\int_q^1 \frac{X_j(p)\,dp}{\sqrt{p^2-q^2}}, \qquad j = 0, 1, 2, ...$$

Hence

$$\int_q^1 |\Phi_i(q) - \Phi_0(q)|\,dq \le \int_{q_0}^1 \left(1 - \frac{2}{\pi}\arcsin\frac{q_0}{p}\right)|X_i(p) - X_0(p)|\,dp <$$

$$< \int_{q_0}^1 |X_i(p) - X_0(p)|\,dp,$$

$i = 1, 2, ...$, which proves the assertion.

2. Theorem 2. For any hodograph Γ there exists a sequence of hodographs Γ_j, $j = 1, 2, ...$, on which the functions $X_j(p)$ and $T_j(p)$ converge respectively to $X(p)$ and $T(p)$ in $L(q_0, 1)$, $0 < q_0 < 1$, and at the same time, for some $a > 0$ and for any $j = 1, 2, ...$, the inequality $V_j(p) \le V(p) - a$ holds for $p \in (0, p_1)$.

3. The proof of the theorem is based on the following lemmas.

Definition 1. Let $f(y)$ be a solution of system (1). Then put $Y_f(p) = \inf\{y,\ pf(y) \ge 1\}$. For $f(y) = u(y)$ we retain $Y(p) = Y_u(p)$.

Definition 2. We shall call the waveguide $j_i = (y_i, \bar{y}_i)$ rectangular if $u(y) = u_i$ for $y \in (y_i, y_i + h_i)$ and $u(y) = p_i^{-1}$ for $y \in (y_i + h_i, \bar{y}_i)$, $i = 1, 2, \ldots, n$.

Lemma 1. For any hodograph Γ there exists a solution $u(y)$ of system (1) all of whose waveguides are rectangular. Such a solution can be constructed if the h_i are chosen sufficiently small (see (1)).

Lemma 2. Let $v(y)$ form rectangular waveguides and be a solution of system (1). Suppose, moreover, that $\overline{D}Y_v(p_i - 0) > -\infty$, $1 \le i \le n$, where $\overline{D}f(x)$ is the upper left derivative:

$$\overline{D}f(x) = \overline{\lim}_{y \to x-0}\frac{f(y) - f(x)}{y - x}.$$

Then $Y_v(p) \equiv V(p)$.

Lemma 3. Consider two solutions $v(y)$ and $w(y)$ of system (1), forming rectangular waveguides. Let $h_i = D_i$ for $v(y)$ and $h_i = d_i$ for $w(y)$, with $D_i > d_i$, $i = 1, 2, \ldots, n$. Then $Y_v(p) \ge Y_w(p) + a$ for $p \in (0, p_1)$, where $a = a(d_i, D_i;\ 1 \le i \le n) > 0$.

4. Proof of Theorem 2. Let $u(y)$ be some solution of system (1). Construct another solution $v_0(y)$ of this system in such a way that its waveguides $j_i = (y_i, \bar{y}_i)$ are rectangular and $Y_{v_0}(p) + a \le Y(p)$ for $p \in (0, p_1)$ and some $a > 0$ (Fig. 2). This is possible by Lemmas 1 and 3.

Choose a decreasing sequence of angles $\alpha_j > 0,\ j = 1, 2, \ldots$, such that $\alpha_j \to 0$ as $j \to \infty$. For each α_j construct a function v_j as follows.

From the points $M_i(\bar{y}_i, v_0(\bar{y}_i))$ draw rays $L_{ij},\ i = 1, 2, \ldots, n$, inclined to the y-axis at the angle α_j. We shall assume that α_1 is so small that the abscissa $\hat{y}_{ij}$ of the point of intersection of L_{ij} with the graph of $v_0(y)$ nearest to M_i belongs to $(\bar{y}_i, y_{i+1}),\ j = 1, 2, \ldots;\ i = 1, 2, \ldots, n;\ y_{n+1} = \infty$. By Lemma 2, $\hat{y}_{ij} > \bar{y}_i$. Put $v_j(y) = v(\bar{y}_i) + k_j(y - \bar{y}_i),\ y \in (\bar{y}_i, \hat{y}_{ij}),\ 1 \le i \le n$, and $v_j(y) = v_0(y)$, $y \notin \bigcup_{i=1}^{n} (\bar{y}_i, \hat{y}_{ij})$, $j = 1, 2, \ldots$, where $k_j = \operatorname{tg} \alpha_j$.

To each function $v_j(y)$ there corresponds the hodograph $\Gamma_j\{2X_j(p),\ 2T_j(p)\}$, $p \in (0, 1)$. Denote by $V_j(p)$ the upper edge $Y_{u_j^v}(p)$, where $u_j^v(y)$ is the set of all solutions of system (1) in which the left-hand sides are replaced by $X_j(p)$ and $T_j(p)$. By Lemma 2, $V_j(p) = Y_{v_j}(p)$.

It is clear that $X_j(p) \to X(p)$ and $T_j(p) \to T(p)$ in $L[q_0, 1]$ as $j \to \infty$, and

$$V_j(p) \le Y_v(p) \le Y(p) - a \le V(p) - a$$

for $p \in (0, p_1)$ and $j = 1, 2, \ldots$.

III. **Stability in the case of a deep source.**

1. **Theorem 3.** Given a hodograph $\Gamma_{10}\{X_{10}(p),\ T_{10}(p)\}$ and a sequence of hodographs $\Gamma_{1i}\{X_{1i}(p),\ T_{1i}(p)\}$, $p \in [0, \mathcal{P})$, $i = 1, 2, \ldots$. Then, if

$$\rho_t(T_{10}, T_{1i}) = \int_0^P |T_{10}(p) - T_{1i}(p)|\, dp \to 0$$

as $i \to \infty$, then also

$$\rho_h(H_0, H_i) = \int_0^{P^{-1}} |H_0(r) - H_i(r)|\, dr + |H_0(\mathcal{P}^{-1}) - H_i(\mathcal{P}^{-1})| \to 0.$$

Proof. Let, everywhere below, $i = 1, 2, \ldots,\ j = 0, 1, 2, \ldots$. Introduce the function $F_j(r)$ by the relation

$$dF_j(r) = \frac{dH_j(r)}{r}, \qquad F_j(0) = 0, \qquad r \in [0, \mathcal{P}^{-1}].$$

It is clear that $F_j(r)$ is a nondecreasing function. Define the distance

$$\rho_f(F_0, F_i) = \int_0^{P^{-1}} |F_0(r) - F_i(r)|\, dr + |F_0(\mathcal{P}^{-1}) - F_i(\mathcal{P}^{-1})|.$$

From the condition and the continuity of the functions $T_{1j}(p)$ it follows that

$$T_{1i}(0) \to T_{10}(0)$$

as $i \to \infty$. Let $|T_{1i}(0) - T_{10}(0)| < C$. From the definition of $F_j(r)$ and the expression

$$T_{1j}(p) = \int_0^{P^{-1}} \frac{dF_j(r)}{\sqrt{1 - p^2 r^2}}, \qquad p \in [0, \mathcal{P}),$$

it follows that

$$T_{1j}(0) = F_j(\mathcal{P}^{-1}).$$

Therefore

$$F_j(r) < T_{10}(0) + C.$$

The functions $F_j(r)$ are monotone and uniformly bounded; therefore they form a compact set.

By what has been proved (see (1)), the mapping F into T is one-to-one. We shall show that it is continuous. It is easily proved that

$$\int_{P-\delta}^{P} T_{1j}(p)\, dp < \varepsilon$$

for arbitrary $\varepsilon > 0$, for sufficiently small δ, for all $j = 0, 1, 2, \ldots.$ From Helly' s theorem and from the equivalence of weak convergence and convergence in mean for continuous functions defined on an interval, we obtain that

$$\int_0^{P-\delta} |T_{1i}(p) - T_{10}(p)|\, dp \to 0$$

when $\rho_f(F_i, F_0) \to 0$. Consequently,

$$\rho_t(T_{1i}, T_{10}) \to 0$$

when $\rho_f(F_i, F_0) \to 0$.

By the theorem on a continuous and one-to-one mapping of compact sets, the mapping is continuous also in the other direction, i.e.

$$\rho_f(F_i, F_0) \to 0$$

when

$$\rho_t(T_{1i}, T_{10}) \to 0.$$

Hence, by Helly' s theorem,

$$|H_0(\mathcal{P}^{-1}) - H_i(\mathcal{P}^{-1})| \to 0$$

when $\rho(T_{1i}, T_{10}) \to 0$.

Finally,

$$H_j(r) = \int_0^r x\, dF_j(x) = rF_j(r) - \int_0^r F_j(x)\, dx,$$

and

$$\int_0^{P^{-1}} |H_i(r) - H_0(r)|\, dr \leq 2\mathcal{P}^{-1} \int_0^{P^{-1}} |F_i(r) - F_0(r)|\, dr.$$

Therefore

$$\rho_h(H_0, H_i) \to 0$$

when

$$\rho_t(T_{10}, T_{1i}) \to 0,$$

which was required to prove.

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Received
11 III 1967

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Note: Figure translations are in progress. See original paper for figures.

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