DIFFERENCE SCHEMES OF THIRD-ORDER ACCURACY FOR THROUGH COMPUTATION OF DISCONTINUOUS SOLUTIONS

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Abstract

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MATHEMATICS

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DIFFERENCE SCHEMES OF THIRD-ORDER ACCURACY FOR THROUGH COMPUTATION OF DISCONTINUOUS SOLUTIONS

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1°. In constructing difference schemes for through computation of gas-dynamic problems with discontinuous solutions, the requirements of high approximation accuracy and good computational quality are, in a certain sense, contradictory. Thus, the requirement that the scheme be monotone restricts the approximation accuracy to first order ([1]), and increasing the order of approximation to second leads to strong oscillations in the vicinity of a discontinuity. For schemes of second-order accuracy, many authors, beginning with ([2]), have used various devices to weaken this phenomenon, based on introducing smoothing terms into the difference scheme that act only in the vicinity of the discontinuity. Schemes of higher order of accuracy, as far as we know, have not been considered for nonlinear hyperbolic systems. Meanwhile it is known ([3]) that, for the equation $w_t = aw_x$, a scheme of third-order accuracy gives substantially smaller oscillations near a discontinuity than a second-order scheme, and the speed of expansion of the zone of "smearing" of the discontinuity is also smaller for it.

This suggests using schemes of third-order accuracy for the computation of discontinuous solutions. Since the equations are written in divergent form, it is expedient to use explicit schemes in order to avoid the need to solve, at each time step, a system of nonlinear equations with a large number of unknowns.

2°. Consider a nonlinear system of first-order equations in divergent form

$$\partial w/\partial t = \partial F(w, x, t)/\partial x + f(w, x, t). \tag{1}$$

We shall assume that, in the domain of values of $w(x, t)$, x, t under consideration, the system is hyperbolic, i.e. the matrix $F_w = \partial F/\partial w$ has real and distinct eigenvalues.

Introduce in the x, t-plane a mesh with steps $\Delta x = h$ and $\Delta t = \tau$, denoting:

$$x_m = mh, \qquad t^n = n\tau, \qquad q = \tau/h$$

$$w(x_m, t^n) = w_m^n, \qquad w^n = \{w_m^n\}.$$

The numerical solution of the Cauchy problem for system (1) reduces to an algorithm for computing w^{n+1} under the condition that w^k for $k \leq n$ are already known. It is clear that every method for the numerical solution of the Cauchy problem for system (1) corresponds to some method for solving the Cauchy problem for a system of ordinary equations

$$w_t = f(w, t). \tag{2}$$

One may expect that, conversely, every method for solving the Cauchy problem for (2) has its generalization for (1).

For system (2), the following methods for computing w^{n+1} are well known: (a) difference methods of Adams type; (b) methods based on expanding the solution in a series; and (c) iterative methods of Runge–

Kutta. Generalizations of the first two methods lead either to multistep schemes (a), or to very cumbersome formulas containing tensors of the third order (b). Both are unacceptable if one has in mind a further generalization to equations with a large number of independent variables.

As for the third case, so far as we know, until now there have been no generalizations of the Runge–Kutta method for systems of partial differential equations. Meanwhile, as it turned out, such a generalization leads to an economical and logically simple iterative scheme of third-order accuracy. The method of constructing the scheme is sufficiently general and admits extension to systems of equations with a large number of independent variables.

3°. Passing to the construction of the scheme, define w^{n+1} as the result of the iterative process

$$\{w_m^n\} = w^n = w^{[0]} \to w^{[1]} \to \cdots \to w^{[R]} = w^{n+1} = \{w_m^{n+1}\}.$$

We shall construct the difference scheme so that the value w_m^{n+1} depends on 5 values w_m^n at the points $m, m \pm 1, m \pm 2$. For a third-order scheme, generally speaking, 4 points are sufficient, but a 5-point scheme is more symmetric and natural for a general system of equations in which the characteristics may be directed arbitrarily.

Introduce notation for the averaging operators, the difference-taking operator, and the identity operator:

$$\mu\varphi_\varepsilon = 2^{-1}(\varphi_{\varepsilon+1/2} + \varphi_{\varepsilon-1/2});$$

$$\delta\varphi_\varepsilon = \varphi_{\varepsilon+1/2} - \varphi_{\varepsilon-1/2}; \qquad I\varphi_\varepsilon = \varphi_\varepsilon,$$

where ε is an integer or half-integer index.

Put $R = 3$ and write in general form the formulas for $w^{[1]}$, $w^{[2]}$, and $w^{[3]}$, imposing on them only the conditions of homogeneity and symmetry:

$$w_m^{[0]} = w_m^n,$$

$$w_{m+1/2}^{[1]} = \mu w_{m+1/2}^{[0]} + \beta_{10}\{q\delta F + \tau\mu f\}_{m+1/2}^{[0]},$$

$$w_m^{[2]} = (I + \omega_{21}\delta^2)w_m^{[0]} + \beta_{20}\{q\mu\delta F + \tau(I + \gamma_{21}\delta^2)f\}_m^{[0]} + \beta_{21}\{q\delta F + \tau\mu f\}_m^{[1]},$$

$$\begin{aligned} w_m^{n+1} = w_m^{[3]} = (I + \omega_{31}\delta^2 + \omega_{32}\delta^4)w_m^{[0]} + \beta_{30}\{q(I + \theta_{31}\delta^2)\mu\delta F \\ + \tau(I + \gamma_{31}\delta^2 + \gamma_{32}\delta^4)f\}_m^{[0]} + \beta_{31}\{q(I + \theta_{32}\delta^2)\delta F \\ + \tau(I + \gamma_{33}\delta^2)\mu f\}_m^{[1]} + \beta_{32}\{q\mu\delta F + \tau(I + \gamma_{34}\delta^2)f\}_m^{[2]}. \end{aligned} \tag{3}$$

Here

$$\{q\mu^k\delta^l F\}_\xi^{[r]} = q\mu^k\delta^l F(w_\xi^{[r]}, x_\xi, t^n + \alpha_r\tau)$$

and similarly for $\{\tau\mu^k\delta^l f\}_\xi^{[r]}$.

The quantities α_r, β_{rs}, ω_{rs}, θ_{rs}, γ_{rs} are constant parameters, which should be chosen so that the expansion of $w_m^{[3]}$ in powers of τ coincides with the expansion of w_m^{n+1} in a Taylor series in t up to and including terms of order τ^3. It is assumed here that τ and h are related by

$$\tau/h = q = \text{const}.$$

After very cumbersome computations and comparison of the expansions, we obtain equations for the parameters. Analysis of them shows that, in contrast to the Runge–Kutta scheme for system (2), for the system in partial derivatives there exists only one difference scheme of third-order accuracy of the form (3). The corresponding values of the parameters are

$$\alpha_1 = \beta_{10} = 1/3; \qquad \alpha_2 = \beta_{21} = 2/3; \qquad \beta_{30} = 1/4; \qquad \beta_{32} = 3/4; \qquad \theta_{31} = -2/3;$$

Fig. 1

Figure 1: Fig. 1

$$\beta_{20} = \beta_{31} = \omega_{21} = \omega_{31} = \gamma_{31} = \gamma_{32} = \gamma_{34} = 0.$$

The coefficient ω_{32} does not affect the accuracy of the approximation, but is essential for ensuring stability. An elementary investigation of stability by the Fourier method with constant coefficients, i.e., for $F_w = \text{const}$, leads to the conditions

$$-3 \leq 24\omega_{32} \leq \sigma^4 - 4\sigma^2,$$

where $\sigma = q|\lambda|_{\max}$, and $|\lambda|_{\max}$ is the maximum modulus of the eigenvalues of the matrix F_w. Numerical verification has shown that these conditions ensure stability also for nonlinear equations.

Of course, the form (3) is not the only one for a third-order scheme; however, it is the simplest in the sense that it does not contain matrix derivatives of the vectors F and f, and therefore the number of operations in formulas (3) is minimal.

Fig. 1

4°. To test the scheme, a number of computations were carried out for discontinuous solutions of the gas-dynamics equations in Lagrangian and Eulerian coordinates.

Figure 1 presents the distributions of pressure p and density ρ, obtained as a result of a computation in Eulerian coordinates of the decay of a discontinuity with the formation of a shock wave on the left and a rarefaction wave on the right. In the computation $q = 0.15$, $\omega_{32} = -0.104$, $n = 120$. The thin lines show the exact distributions.

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Note: Figure translations are in progress. See original paper for figures.