Transformative Semigroups \((\mathfrak{G},\circ,\xi,\chi_1,\ch

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Abstract

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Transformative Semigroups $(\mathfrak{G}, \circ, \xi, \chi_1, \chi_2)$

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It is known that not all properties of (partial) transformations of a set can be expressed in terms of the operation of their multiplication. Consequently, in considering an abstract semigroup isomorphic to a given semigroup of transformations, we cannot recover many important connections between the corresponding transformations. Among such connections are the most important binary relations defined in semigroups of transformations: the projection relations χ_1 (first) and χ_2 (second) ($(\varphi_1, \varphi_2) \in \chi_i$ if and only if $pr_i\varphi_1 \subset pr_i\varphi_2$), and the relation of semicompatibility of transformations ξ ($(\varphi_1, \varphi_2) \in \xi$ if and only if φ_1 and φ_2 act identically on $pr_1\varphi_1 \cap pr_1\varphi_2$; for transformations their compatibility is usually denoted by an infix sign).

The desire to give as complete an abstract description as possible of semigroups of transformations leads to the natural concept of a relativized semigroup [3]: a system $(\mathfrak{G}, \circ, \rho_1, \dots, \rho_n)$, where $(\mathfrak{G}, \circ)$ is a semigroup, and $\rho_1, \dots, \rho_n$ are certain binary relations given on $\mathfrak{G}$. If $(\mathfrak{G}, \circ, \rho_1, \dots, \rho_n)$ and $(\mathfrak{G}^*, \circ^*, \rho_1^*, \dots, \rho_n^*)$ are two relativized semigroups, then a mapping $P : \mathfrak{G} \to \mathfrak{G}^*$ is called a homomorphism of the first into the second if P is a homomorphism of $(\mathfrak{G}, \circ)$ into $(\mathfrak{G}^*, \circ^*)$, and, moreover, $(g_1, g_2) \in \rho_i$ if and only if $(P(g_1), P(g_2)) \in \rho_i^*$. The study of relativized semigroups of transformations was initiated by V. V. Wagner [1]. He also posed the problem of an abstract characterization of relativized semigroups of transformations $(\mathfrak{G}, \circ, \xi, \chi_1, \chi_2)$.

Relativized semigroups isomorphic to relativized semigroups of transformations of the form $(\mathfrak{G}, \circ, \xi, \chi_1, \chi_2)$ will be called **transformative semigroups**. Attempts to describe the class of transformative semigroups led to the isolation of the classes of transformative semigroups of the first kind $(\mathfrak{G}, \circ, \xi, \chi_1)$ [2], projection semigroups of the first and second kinds $(\mathfrak{G}, \circ, \chi_1)$ and $(\mathfrak{G}, \circ, \chi_2)$, respectively [5], general projection semigroups $(\mathfrak{G}, \circ, \chi_1, \chi_2)$ [3], and relativized semigroups $(\mathfrak{G}, \circ, \xi)$ [6]. In the present note an elementary (but infinite) system of axioms is given for transformative semigroups $(\mathfrak{G}, \circ, \xi, \chi_1, \chi_2)$. The problem is solved by the same method with whose aid the author obtained an abstract characterization of projection semigroups [3] and an elementary axiomatics of transformative semigroups of the first kind [4].

Let $(\mathfrak{G}, \circ, \xi, \chi_1, \chi_2)$ be some relativized semigroup. Define a ternary predicate Π as follows. Put

$$\Pi_0(g_1, g_2, g_3) \leftrightarrow (\exists \bar{g}_1, \bar{g}_2, \dots, \bar{g}_{m-1}, \bar{g}_m, g)((\bar{g}_1, \bar{g}_2) \in \xi \wedge \dots \wedge (\bar{g}_{m-1}, \bar{g}_m) \in \xi \wedge \{\bar{g}_1 g, \bar{g}_2, \dots, \bar{g}_{m-1}, \bar{g}_m\} \subset \chi_1\langle g_1\rangle \cup \chi_1\langle g$$

$$\Pi_n(g_1, g_2, g_3) \leftrightarrow (\exists \bar{g}_1, \bar{g}_2, \dots, \bar{g}_{m-1}, \bar{g}_m, g)((\bar{g}_1, \bar{g}_2) \in$$

$$\in \xi \wedge \dots \wedge (\bar{g}_{m-1}, \bar{\bar{g}}_m) \in \xi \wedge \Pi_{n-\bar{g}_1}(g_1, g_2, \bar{g}_1 g) \wedge \Pi_{n-1}(g_1, g_2, \bar{g}_2) \wedge \dots$$
$$\dots \wedge \Pi_{n-1}(g_1, g_2, \bar{g}_m) \wedge (\bar{g}_m g, g_3) \in \chi_1;$$
$$\Pi(g_1, g_2, g_3) \leftrightarrow (\exists n)\, \Pi_n(g_1, g_2, g_3)$$

(the symbol g may be empty).

We construct inductively, over the alphabet with letters $a_g, b_g, c_{(g_1, g_2)}$ $(g, g_1, g_2 \in \mathfrak{G})$, the following set of words Γ.

First Γ_1 is defined:

1. $c_{(g_1, g_2)} \in \Gamma_1$.
2. $\Pi(g_1, g_2, g_3) \to c_{(g_1, g_2)} b_{g_3} \in \Gamma_1$.
3. $\Pi(g_1, g_2, g_3) \wedge (\exists \bar{g}_3, \dots, \bar{\bar{g}}_3, \tilde{g}_3)((g_3, \bar{g}_3) \in \xi \wedge \dots \wedge (\bar{\bar{g}}_3, \tilde{g}_3) \in \xi \wedge$
$$\wedge \Pi(g_1, g_2, \bar{g}_3) \wedge \dots \wedge \Pi(g_1, g_2, \bar{\bar{g}}_3) \wedge (\tilde{g}_3, g_4) \in \chi_2 \to c_{(g_1, g_2)} b_{g_3} a_{g_4} \in \Gamma_1.$$
4. $\gamma b_{g_1} a_{g_2} \in \Gamma_1 \wedge (\exists \bar{g}_2, \bar{\bar{g}}_2, \bar{g}_3, \bar{\bar{g}}_3, g)((\bar{g}_3 g, g_3) \in \chi_1 \wedge (g_2 \bar{g}_2, \bar{g}_3) \in$
$$\in \chi_1 \wedge \gamma b_{g_1 \bar{g}_2} \in \Gamma_1 \wedge (\bar{g}_2 \bar{\bar{g}}_2, \bar{\bar{g}}_3 g) \in \chi_1 \wedge \gamma b_{g_1 \bar{\bar{g}} 2} \in \Gamma_1 \wedge (\bar{\bar{g}}_3, \bar{g}_3) \in \xi) \to$$
$$\to \gamma b_{g_1} a_{g_2} b_{g_3} \in \Gamma_1$$
($\bar{g}_2, \bar{\bar{g}}_2, g$ may be empty symbols).
5. $\gamma b_{g_1} a_{g_2} b_{g_3} \in \Gamma_1 \wedge (\exists \tilde{g}_3, \dots, \bar{\bar{g}}_3, \bar{g}_3)((g_3, \tilde{g}_3) \in \xi \wedge \dots \wedge (\bar{\bar{g}}_3, \bar{g}_3) \in \xi \wedge$
$$\wedge \gamma b_{g_1} a_{g_2} b_{\tilde{g}_3} \in \Gamma_2 \wedge \dots \wedge \gamma b_{g_1} a_{g_2} b_{\bar{\bar{g}}_3} \in \Gamma_1 \wedge ((\bar{g}_3, g_4) \in \chi_2 \vee (\exists \bar{g}_2)(\bar{g}_3 = g_2 \bar{g}_2 \wedge$$
$$\wedge \gamma b_{g_1 \bar{g}_2} a_{g_4} \in \Gamma_1))) \to \gamma b_{g_1} a_{g_2} b_{g_3} a_{g_4} \in \Gamma_1$$
($\bar{g}_2$ may be an empty symbol).
6. There are no other words in Γ_1.

Suppose now that Γ_n has been constructed. Then:

1. $\gamma b_{g_1} a_{g_2} b_{g_3} \in \Gamma_n \wedge g_3 = g_2 \bar{g}_2 \to \gamma b_{g_1 \bar{g}_2} \in \Gamma_{n+1}$ ($\bar{g}_2$ may be empty).

2. $\gamma b_{g_1} a_{g_2} b_{g_3} a_{g_4} \in \Gamma_n \wedge g_3 = g_2 \bar{g}_2 \to \gamma b_{g_1 \bar{g}_2} a_{g_4} \in \Gamma_{n+1}$ ($\bar{g}_2$ may be empty).
3. $c_{(g_1,g_2)} b_{g_3} \in \Gamma_{n+1} \wedge (\exists g_3, \dots, \bar{\bar{g}}_3, \tilde{g}_3)((g_3, \bar{g}_3) \in \xi \wedge \dots \wedge (\bar{\bar{g}}_3, \tilde{g}_3) \in$

$$\in \xi \wedge c_{(g_1,g_2)} b_{\bar{g}_3} \in \Gamma_{n+1} \wedge \dots \wedge c_{(g_1,g_2)} b_{\bar{\bar{g}}3} \in \Gamma_{n+1} \wedge (\tilde{g}_3, g_4) \in \chi_2) \to$$

$$\to c_{(g_1,g_2)} b_{g_3} a_{g_4} \in \Gamma_{n+1}.$$

4–5. As in the definition of Γ_1, with Γ_1 replaced everywhere by Γ_{n+1}.

6. There are no other words in Γ_{n+1}.

Finally, put

$$\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n.$$

On the set Γ define a binary relation ε_1, setting $(\gamma_1, \gamma_2) \in \varepsilon_1$ if and only if

$$\gamma_1 = \gamma b_{g_1} a_{g_2} b_{g_3} \dots a_{g_{2n}} b_{g_{2n+1}} (a_{g_{2n+2}}), \quad \gamma_2 = \gamma b_{\tilde{g}_1} a_{g_2} b_{\tilde{g}_3} \dots a_{g_{2n}} b_{\tilde{g}_{2n+1}} (a_{g_{2n+2}})$$

and $(g_{2k+1}, \tilde{g}_{2k+1}) \in \xi$ $(k = 0, 1, \dots, n)$.

Lemma 1. If $\mathbf{I}\Delta_{\mathfrak{G}} \subset \xi$ and $\Pi_{\xi}^{-1} \circ \subset \xi$ (i.e. ξ is reflexive and symmetric), then

$$\bar{\varepsilon}_1 = \bigcup_{n=1}^{\infty} \varepsilon_1^n$$

is an equivalence relation.

Using $\bar{\varepsilon}_1$, we introduce in Γ one more binary relation. Namely, let

$$\gamma_1 = \bar{\gamma} b_{g_1} a_{g_2} b_{g_3} \dots b_{g_{2n-1}} a_{g_{2n}} b_{g_{2n+1}} \bar{\bar{\gamma}},$$

$$\gamma_2 = \tilde{\gamma} b_{g_1} a_{g_2} b_{g_3} \dots b_{g_{2m-1}} a_{g_{2m}} b_{g_{2m+1}} \bar{\gamma} \quad \text{and} \quad g_{2n+1} = g_{2n} \tilde{\delta}_{2n}, \quad g_{2n-1} \tilde{\delta}_{2n}$$

$$= g_{2n-2} \tilde{\delta}_{2n-2}, \dots, \quad g_3 \tilde{\delta}_4 = g_2 \tilde{\delta}_2, \quad \bar{g}_{2m+1} = \bar{g}_{2m} \bar{\delta}_{2m}, \quad \bar{g}_{2m-1} \bar{\delta}_{2m}$$

$$= \bar{g}_{2m-2} \bar{\delta}_{2m-2}, \dots, \quad \bar{g}_3 \bar{\delta}_4 = \bar{g}_2 \bar{\delta}_2$$

$$(\tilde{\delta}_{2n}, \tilde{\delta}_{2n-2}, \dots, \tilde{\delta}_4, \tilde{\delta}_2, \bar{\delta}_{2m}, \bar{\delta}_{2m-2}, \dots, \bar{\delta}_4, g_2$$

may be empty symbols),

$$(\bar{\gamma} b_{g_1 \bar{g}2} \gamma, \gamma b_{g_1 \bar{g}2} \bar{\gamma}) \in \bar{\varepsilon}_1.$$

In this and only in this case we regard $(\gamma_1, \gamma_2) \in \bar{\varepsilon}_2$. Putting

$$\bar{\varepsilon}_2 = \bigcup_{n=0}^{\infty} \varepsilon_2^n,$$

we construct the fundamental relation on the set Γ:

$$\varepsilon = \bigcup_{n=1}^{\infty} \bar{\varepsilon}_2 \circ \bar{\varepsilon}_1 \ \cup \ \bar{\varepsilon}_1 \circ \bar{\varepsilon}_2.$$

By Lemma 1 and the definition of $\bar{\varepsilon}_2$, ε will be an equivalence relation. ε-equivalent words will be regarded as different writings of one and the same word.

Introduce the notation

$$\gamma \circ b_g = \begin{cases} \gamma_0 b_{g_1} a_{g_2} b_g, & \text{if } \gamma = \gamma_0 b_{g_1} a_{g_2}, \\ \gamma_0 b_{g_1 g}, & \text{if } \gamma = \gamma_0 b_{g_1}, \\ c_{(g_1, g_2)} b_g, & \text{if } \gamma = c_{(g_1, g_2)}; \end{cases}$$

and $\gamma \circ a_g = \gamma_0 b_{g_1} a_g$, if $\gamma = \gamma_0 b_{g_1}$.

Lemma 2. If

$$\text{III} \quad (g_1, g_2) \in \xi \ \wedge \ (g_3, g_4) \in \xi \to (g_1 g_3, g_2 g_4) \in \xi$$

(i.e. ξ is stable), then from $(\gamma_1, \gamma_2) \in \varepsilon$ it follows, under the condition $(g_1, g_2) \in \xi$, that

$$(\gamma_1 \circ b_{g_1}, \gamma_2 \circ b_{g_2}) \in \varepsilon,$$

if both these words belong to Γ.

Now put $(\gamma_1, \gamma_2) \in \omega$ if and only if:
1. $\gamma_2 = c_{(g_2, g_2)}$ and $\gamma_1 \circ b_{g_2} \in \Gamma$.
2. $\gamma_2 = c_{(g_2, g_2)} b_{g_2}$ and $\gamma_1 \circ a_{g_2} \in \Gamma$.

Lemma 3. If:

$$\text{IV} \quad c_{(g_1, g_1)} b_{g_2} \in \Gamma \to (g_1, g_2) \in \chi_1,$$

$$\text{V} \quad \chi_1^2 \subset \chi_1 \quad (\text{i.e. } \chi_1 \text{ is transitive}),$$

$$\text{VI} \quad (g_1, g_2) \in \chi_1 \to (g_3 g_1, g_3 g_2) \in \chi_1 \quad (\text{i.e. } \chi_1 \text{ is left regular}),$$

$$\text{VII} \quad c_{(g_1, g_1)} b_{g_1} a_{g_2} \in \Gamma \to (g_1, g_2) \in \chi_2,$$

$$\text{VIII} \quad \chi_2^2 \subset \chi_2 \quad (\text{i.e. } \chi_2 \text{ is transitive}),$$

then the binary relation ω is transitive.

Proof. Let $(\gamma_1, \gamma) \in \omega$ and $(\gamma, \gamma_2) \in \omega$.

1. $\gamma_2 = c_{(g_2, g_2)}$, $\gamma = c_{(g, g)}$, and $c_{(g,g)} b_{g_2} \in \Gamma$. According to IV, $(g, g_2) \in \chi_1$;

a) $\gamma_1 = \gamma_0 a_{g_1}$, $\gamma_0 a_{g_1} b_g \in \Gamma$. By V and the definition of Γ, $\gamma_0 a_{g_1} b_{g_2} \in \Gamma$;

b) $\gamma_1 = \gamma_0 b_{g_1}$, $\gamma_0 b_{g_1 g} \in \Gamma$. In view of VI, $(g_1 g, g_1 g_2) \in \chi_1$, whence $(\gamma_1, \gamma_2) \in \omega$.

2. $\gamma_2 = c_{(g_2,g_2)}$, $(\gamma, c_{(g,g)} b_g) \in \varepsilon$, $\gamma \circ b_{g_2} \in \Gamma$. By Lemma 2,

$$(\gamma \circ b_{g_2},\ c_{(g,g)} b_g \circ b_{g_2}) \in \varepsilon.$$

Here

$$c_{(g,g)} b_{gg_2} = c_{(g,g)} b_g \circ b_{g_2}$$

belongs to Γ together with $\gamma \circ b_{g_2}$, by the definition of Γ. Then IV gives $(g, gg_2) \in \chi_1$. If $\gamma_1 = \gamma_0 b_{g_1}$, then, by the condition, $\gamma_0 b_{g_1} a_g \in \Gamma$. Obviously,

$$\gamma_0 b_{g_1} a_g b_{gg_2} \in \Gamma,$$

and then also

$$\gamma_0 b_{g_1} \circ b_{g_2} = \gamma_0 b_{g_1 g_2} \in \Gamma.$$

3. $\gamma_2 = c_{(g_2,g_2)} b_{g_2}$, $(\gamma, c_{(g,g)} b_g) \in \varepsilon$, $\gamma \circ a_{g_2} \in \Gamma$. Then

$$c_{(g,g)} b_g a_{g_2} \in \Gamma,$$

i.e., by VII, $(g_1, g_2) \in \chi_2$. Since $\gamma_1 = \gamma_0 b_{g_1}$, from $\gamma_0 b_{g_1} a_g \in \Gamma$, according to VIII, we obtain

$$\gamma_0 b_{g_1} a_{g_2} \in \Gamma.$$

The lemma is proved.

Lemma 4. If

$$\text{IX} \quad (g_1 g_2, g_1) \in \chi$$

(i.e. χ is right-negative), then

$$\gamma \in \omega^{-1}\langle c_{(g_1 g_2, g_1 g_2)}\rangle \leftrightarrow \gamma \in \omega^{-1}\langle c_{(g_1,g_1)}\rangle \ \wedge\ \gamma \circ b_{g_1} \in \omega^{-1}\langle c_{(g_2,g_2)}\rangle.$$

Lemma 5. If

$$\text{X} \quad \Delta_{\mathfrak{G}} \subset \chi_2 \quad \text{(i.e. } \chi_2 \text{ is reflexive)},$$

$$\text{XI} \quad (g_1 g_2, g_2) \in \chi_2 \quad \text{(i.e. } \chi_2 \text{ is left-negative)},$$

$$\text{XII} \quad \Delta_{\mathfrak{G}} \subset \chi_1 \quad \text{(i.e. } \chi_1 \text{ is reflexive)},$$

then

$$\omega^{-1}\langle c_{(g,g)}\rangle \circ b_g = \omega^{-1}\langle c_{g,g} b_g\rangle.$$

Proof. Let $\gamma_0 \in \omega^{-1}\langle c_{(g,g)} \circ b_g\rangle$, and moreover $\gamma_0 = \gamma_0 \circ b_g$. If $\gamma_0 = \gamma_0 \circ a_{g_0}$, then $\gamma \in \omega\langle c_{(g,g)} b_g\rangle$ by X; and if $\gamma_0 = \gamma_0 b_{g_0}$, then we apply XI. Conversely, assuming $\gamma \in \omega\langle c_{(g,g)} b_g\rangle$, i.e. $\gamma \circ a_g \in \Gamma$, we have, in view of XII, $\gamma a_g b_g \in \Gamma$, but $\gamma a_g b_g = \gamma a_g \circ b_g$, and $\gamma a_g \in \omega^{-1}\langle c_{(g,g)}\rangle$, and $(\gamma, \gamma a_g b_g) \in \varepsilon$.

Theorem 1. *If in a relativized semigroup* $(\mathfrak{G}, \circ, \xi, \chi_1, \chi_2)$ *the binary relations* ξ, χ_1, χ_2 *satisfy conditions I–XII and XIII*

$$(c_{(g_1,g_2)}b_{g_1},\, c_{(g_1,g_2)}b_{g_2}) \in \varepsilon \to (g_1, g_2) \in \xi,$$

then it admits a homomorphism into some transformative semigroup of transformations.

Proof. Construct a mapping P of the set $\mathfrak{G}$ into the set $\mathfrak{F}(\Gamma \times \Gamma)$ of all transformations of Γ, putting

$$pr_1 P(g) = \omega^{-1}\langle c_{(g,g)} \rangle$$

and

$$P(g)(\gamma) = \gamma \circ b_g.$$

Lemma 4 and the stability of ξ (condition III) ensure the equality

$$P(g_1 g_2) = P(g_2) \circ P(g_1)$$

(we recall that equality of words from Γ is understood in the sense of their ε-equivalence). Further,

$$pr_1 P(g_1) \subset pr_1 P(g_2)$$

is equivalent, by definition, to the inclusion

$$\omega^{-1}\langle c_{(g_1,g_1)} \rangle \subset \omega^{-1}\langle c_{(g_2,g_2)} \rangle,$$

which, in turn, in view of Lemma 3 and the reflexivity of χ_2 (condition XII), is replaced by the relation

$$c_{(g_1,g_1)} \in \omega^{-1}\langle c_{(g_2,g_2)} \rangle,$$

i.e.

$$c_{(g_1,g_1)} b_{g_2} \in \Gamma.$$

But the latter, according to IV, is equivalent to

$$(g_1, g_2) \in \chi_1.$$

Analogously, Lemmas 4 and 5, the reflexivity of χ_2 (condition X), and condition VII ensure the equivalence of the relations

$$pr_2 P(g_1) \subset pr_2 P(g_2)$$

and

$$(g_1, g_2) \in \chi_2.$$

Let now $(g_1, g_2) \in \xi$. Then from Lemma 2 it follows that, if

$$\gamma \in pr_1 P(g_1) \cap pr_1 P(g_2),$$

then

$$(P(g_1)(\gamma),\, P(g_2)(\gamma)) \in \varepsilon,$$

i.e. $P(g_1)P(g_2)$. Conversely, if $P(g_1)$ and $P(g_2)$ are half-compatible, then, in particular,

$$P(g_1)(c_{(g_1,g_2)}) = P(g_2)(c_{(g_1,g_2)}),$$

i.e.

$$(c_{(g_1,g_2)}b_{g_1},\, c_{(g_1,g_2)}b_{g_2}) \in \varepsilon,$$

whence, by XIII, $(g_1, g_2) \in \xi$. The theorem is proved.

Theorem 2. *In order that the relativized semigroup* $(\mathfrak{G}, \circ, \xi, \chi_1, \chi_2)$ *be transformative, it is necessary and sufficient that the binary relations* ξ, χ_1, χ_2 *satisfy conditions I–XIII and XIV*

$$\xi \cap \chi_1 \cap \chi_1^{-1} \subset \Delta_{\mathfrak{G}}.$$

Proof. Sufficiency. Under condition XIV the mapping P defined in the preceding theorem will be one-to-one. Indeed, if

$$P(g_1) = P(g_2),$$

then

$$(g_1, g_2) \in \chi_1 \cap \chi_1^{-1}$$

(since

$$pr_1 P(g_1) = pr_1 P(g_2)$$

) and

$$(g_1, g_2) \in \xi$$

(since $P(g_1)P(g_2)$). Hence

$$g_1 = g_2$$

according to XIV.

The **necessity** of all the conditions except IV, VIII, XIII is known ([3,4]). The proof of the necessity of IV, VIII, XIII involves certain technical difficulties.

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Note: Figure translations are in progress. See original paper for figures.