A SPHERICAL CUT IN AN ELASTIC SPACE

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Fig. 1. Spherical cut in space

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Abstract

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THEORY OF ELASTICITY

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A SPHERICAL CUT IN AN ELASTIC SPACE

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In the present paper an axially symmetric problem of the theory of elasticity is solved in quadratures for a space cut along part of a spherical surface. Apparently, this problem is considered for the first time.

1. In a space with a cut located on the surface of a sphere of radius R, we introduce a cylindrical coordinate system z, r, θ, where z is the axis of rotation. We take the expressions for the components of the displacement vector in the Trefftz form ([1])

$$\begin{aligned} 2Gw &= \varphi_z + z\,\partial\psi/\partial z; \\ 2Gu &= \varphi_r + z\,\partial\psi/\partial r. \end{aligned} \tag{1}$$

Here G is the shear modulus; $\varphi_z, \varphi_r, \psi$ are real functions of the variables z, r, satisfying the equations

$$\Delta\varphi_z = 0; \qquad \Delta(\varphi_r e^{i\theta}) = 0; \qquad \Delta\psi = 0$$

$$\left(\Delta = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)$$

and connected by the relation

$$\partial\varphi_r/\partial r + \varphi_r/r + \partial\varphi_z/\partial z + (3-4\nu)\partial\psi/\partial z = 0, \tag{2}$$

where ν is Poisson' s ratio.

Fig. 1. Spherical cut in space

Following works ([2,3]), in the plane zor we introduce analytic functions of the complex variable $\Phi_1(\zeta), \Phi_2(\zeta), \Phi_3(\zeta)$ by the relations

$$\psi(z,r) = \frac{1}{\pi i}\int_{\bar t}^{t} \frac{\Phi_1(\zeta)\,d\zeta}{\sqrt{(\zeta-t)(\zeta-\bar t)}}; \qquad \varphi_z(z,r) = \frac{1}{\pi i}\int_{\bar t}^{t} \frac{\Phi_2(\zeta)\,d\zeta}{\sqrt{(\zeta-t)(\zeta-\bar t)}}; \quad (3)$$

$$\varphi_r(z,r) = \frac{1}{\pi i r}\int_{\bar t}^{t} \frac{\Phi_3(\zeta)(\zeta-z)\,d\zeta}{\sqrt{(\zeta-t)(\zeta-\bar t)}} \qquad (t = z+ir,\ \bar t = z-ir).$$

The functions $\Phi_k(\zeta)$ satisfy the condition $\Phi_k(\bar\zeta) = \overline{\Phi_k(\zeta)}$, $k = 1,2,3$. The contour connecting the points t and $\bar t$ in formulas (3) intersects the axis oz and is such that the function $\sqrt{(\zeta-t)(\zeta-\bar t)}$ remains throughout on one branch. We assume that when $t = Re^{i\alpha}$ and $\zeta = Re^{i\theta}$ lie on L^+, then $\arg\sqrt{(\zeta-t)(\zeta-\bar t)} \to 0$ as $\alpha \to 0,\ \theta \to 0$ $(\alpha > \theta)$.

Substituting (3) into (2), we arrive at an equality which must hold everywhere in the region,

$$\Phi_3'(\zeta) = (3-4\nu)\Phi_1'(\zeta) + \Phi_2'(\zeta). \quad (4)$$

Substituting (3) into (1), taking (4) into account, we obtain

$$2Gu = \frac{1}{\pi i r}\int_{\bar t}^{t} \{\Phi_2(\zeta) + (3-4\nu)\Phi_1(\zeta) + z\Phi_1'(\zeta)\} \frac{(\zeta-z)\,d\zeta}{\sqrt{(\zeta-t)(\zeta-\bar t)}}; \quad (5)$$

$$2Gw = \frac{1}{\pi i}\int_{\bar t}^{t} [\Phi_2(\xi) + z\Phi_1'(\xi)] \frac{d\xi}{\sqrt{(\xi-t)(\xi-\bar t)}}.$$

The displacements u and w will vanish at infinity if, for large $|\xi|$,

$$\Phi_1(\xi) = \frac{c_{-1}}{\xi} + \frac{c_{-2}}{\xi^2} + \dots; \qquad \Phi_2(\xi) = \frac{d_{-1}}{\xi} + \frac{d_{-2}}{\xi^2} + \dots,$$

where $(3-4\nu)c_{-1} + d_{-1} = 0$.

Substituting (5) into the known relations connecting stresses and displacements in cylindrical coordinates, we obtain expressions for the stresses:

$$\sigma_z = \frac{1}{\pi i}\int_{\bar t}^{t} [(1-2\nu)\Phi_1'(\xi) + z\Phi_1''(\xi) + \Phi_2'(\xi)] \frac{d\xi}{\sqrt{(\xi-t)(\xi-\bar t)}};$$

$$\tau_{zr} = \frac{1}{\pi i r} \int_{\bar{t}}^{t} [2(1-\nu)\Phi_1'(\xi) + z\Phi_1''(\xi) + \Phi_2'(\xi)] \frac{(\xi - z)\, d\xi}{\sqrt{(\xi - t)(\xi - \bar{t})}};$$

$$\sigma_r = -\frac{1}{\pi i} \int_{\bar{t}}^{t} [(3-2\nu)\Phi_1'(\xi) + z\Phi_1''(\xi) + \Phi_2'(\xi)] \frac{d\xi}{\sqrt{(\xi - t)(\xi - \bar{t})}} -$$

$$-\frac{1}{\pi i r^2} \int_{\bar{t}}^{t} [(3-4\nu)\Phi_1(\xi) + z\Phi_1'(\xi) + \Phi_2(\xi)] \frac{(\xi - z)\, d\xi}{\sqrt{(\xi - t)(\xi - \bar{t})}}; \qquad (6)$$

$$\sigma_\theta = -\frac{1}{\pi i} \int_{\bar{t}}^{t} \frac{2\nu \Phi_1'(\xi)\, d\xi}{\sqrt{(\xi - t)(\xi - \bar{t})}} +$$

$$+\frac{1}{\pi i r^2} \int_{\bar{t}}^{t} [\Phi_2(\xi) + (3-4\nu)\Phi_1(\xi) + z\Phi_1'(\xi)] \frac{(\xi - z)\, d\xi}{\sqrt{(\xi - t)(\xi - \bar{t})}}.$$

2. In the case of the first fundamental problem, the following expressions are prescribed on the boundary:

$$p_z = \sigma_z \cos(n, z) + \tau_{zr} \cos(n, r);$$

$$p_r = \tau_{zr} \cos(n, z) + \sigma_r \cos(n, r), \qquad (7)$$

where n is the outward normal to the boundary surface.

Substituting (6) into (7), we obtain the following expressions on the boundary:

$$R p_z^+(\alpha) = \frac{2}{\pi} \int_0^\alpha \frac{\operatorname{Re}\{A^+(\sigma) e^{i\theta/2}\}\, d\theta}{\sqrt{2(\cos\theta - \cos\alpha)}};$$

$$R p_z^-(\alpha) = \frac{2}{\pi} \int_0^\alpha \frac{\operatorname{Re}\{A^-(\sigma) e^{i\theta/2}\}\, d\theta}{\sqrt{2(\cos\theta - \cos\alpha)}}; \qquad (8)$$

$$-R \sin\alpha\, p_r^+(\alpha) = \frac{2}{\pi} \int_0^\alpha \operatorname{Re}\{B^+(\sigma) e^{i\theta/2}\} \sqrt{2(\cos\theta - \cos\alpha)}\, d\theta;$$

$$-R \sin\alpha\, p_r^-(\alpha) = \frac{2}{\pi} \int_0^\alpha \operatorname{Re}\{B^-(\sigma) e^{i\theta/2}\} \sqrt{2(\cos\theta - \cos\alpha)}\, d\theta,$$

where $\sigma = Re^{i\theta}$ is a point of the contour L;

$$A(\xi) = \xi(R^2 + \xi^2)F''(\xi) + (5 - 4\nu)\xi^2 F'(\xi) + \xi(1 - 2\nu)F(\xi) + \xi\Phi_2'(\xi);$$

$$B(\xi) = \xi^2(R^2 + \xi^2)F'''(\xi) + \xi[R^2 + \xi^2(10 - 4\nu)]F''(\xi) +$$

$$+(15 - 6\nu)\xi^2 F'(\xi) + 3\xi F(\xi) + \xi^2\Phi_2''(\xi) + \xi\Phi_2'(\xi); \qquad \Phi_1'(\xi) = 2\xi F'(\xi) + F(\xi). \quad (9)$$

From the condition of boundedness of the displacements at the end of the cut L it follows that, in the neighborhood of the points $\xi = Re^{i\theta_1}$, $\xi = Re^{-i\theta_1}$, for $A(\xi)$ and $B(\xi)$ the inequalities

$$|A(\xi)| \le \frac{K}{|\xi - c|^\delta}, \quad \delta < \frac{3}{2}; \qquad |B(\xi)| \le \frac{K_1}{|\xi - c|^\delta}, \quad \delta < \frac{5}{2},$$

must hold, where K and K_1 are certain positive constants, and c denotes either end of the cut L.

From (8) we obtain the boundary conditions for $A(\xi)$ and $B(\xi)$:

$$\mathrm{Re}\{A^+(\sigma)e^{i\theta/2}\} = g_1(\theta), \qquad \mathrm{Re}\{B^+(\sigma)e^{i\theta/2}\} = -h_1(\theta),$$

$$\mathrm{Re}\{A^-(\sigma)e^{i\theta/2}\} = g_2(\theta); \qquad \mathrm{Re}\{B^-(\sigma)e^{i\theta/2}\} = -h_2(\theta). \quad (10)$$

Here

$$g_1(\theta) = \varphi(Rp_z^+), \qquad g_2(\theta) = \varphi(Rp_z^-),$$

where

$$\varphi(p) = \frac{d}{d\theta}\int_0^\theta \frac{p(\alpha)\sin\alpha\, d\alpha}{\sqrt{2(\cos\alpha - \cos\theta)}};$$

$$h_1(\theta) = \psi(Rp_r^+), \qquad h_2(\theta) = \psi(Rp_r^-),$$

where

$$\psi(p) = \frac{d}{d\theta}\left[\frac{1}{\sin\theta}\cdot\frac{d}{d\theta}\int_0^\theta \frac{p(\alpha)\sin^2\alpha\, d\alpha}{\sqrt{2(\cos\alpha - \cos\theta)}}\right].$$

Solving, by known methods ([4]), problems (10), we find $A(\xi)$ and $B(\xi)$

$$A(\xi) = \frac{1}{2\pi i}\frac{1}{X_1(\xi)}\int_L g_3(\theta)e^{-i\theta/2}X_1^+(\sigma)\frac{\sigma+\xi}{\sigma-\xi}\frac{d\sigma}{\sigma}+$$

$$+\frac{1}{2\pi i}\frac{1}{X_2(\xi)}\int_L g_4(\theta)e^{-i\theta/2}X_2^+(\sigma)\frac{\xi^2(R-\sigma)+\xi\sigma(R+\sigma)}{R\sigma^2(\sigma-\xi)}\,d\sigma-$$

$$-\frac{1}{2\pi i}\frac{\xi-R}{X_2(\xi)}\int_L g_3(\theta)e^{-i\theta/2}X_1^+(\sigma)\frac{d\sigma}{\sigma};$$

$$B(\xi) = \frac{1}{2\pi i}\frac{1}{X_3(\xi)}\int_L h_3(\theta)e^{-i\theta/2}X_3^+(\sigma)\frac{\xi^2\sigma+\xi R^2+\sigma^3-R^2\sigma}{\sigma^3(\sigma-\xi)}\,d\sigma+$$

$$+\frac{1}{2\pi i}\frac{1}{X_4(\xi)}\int_L h_4(\theta)e^{-i\theta/2}X_4^+(\sigma)\frac{\xi^2\sigma(\sigma+R)+\xi R(R^2-\sigma^2)+\sigma(\sigma^3-R^3)}{\sigma^4(\sigma-\xi)}\,d\sigma-$$

$$-\frac{1}{2\pi i}\frac{\xi(\xi-R)}{X_4(\xi)}\int_L h_3(\theta)e^{-i\theta/2}X_3^+(\sigma)\frac{d\sigma}{\sigma^2}+\frac{a_0\xi(\xi-R)}{X_4(\xi)}.$$

Here

$$X_1(\xi) = \sqrt{(\xi-Re^{i\theta_1})(\xi-Re^{-i\theta_1})};$$

$$X_2(\xi) = [X_1(\xi)]^2; \qquad X_3(\xi) = [X_1(\xi)]^3; \qquad X_4(\xi) = [X_1(\xi)]^4;$$

a_0 is a real constant;

$$2g_3(\theta) = g_1^*(\theta) + g_2^*(\theta), \qquad 2g_4(\theta) = g_1^*(\theta) - g_2^*(\theta),$$

where

$$g_n^*(\theta) = \begin{cases} g_n(\theta), & 0 \le \theta \le \theta_1, \\ g_n(-\theta), & -\theta_1 \le \theta \le 0; \end{cases} \qquad (n = 1, 2)$$

$$2h_3(\theta) = -h_1^*(\theta) - h_2^*(\theta), \qquad 2h_4(\theta) = -h_1^*(\theta) + h_2^*(\theta),$$

where

$$h_n^*(\theta) = \begin{cases} h_n(\theta), & -0 \leqslant \theta \leqslant \theta_1, \\ h_n(-\theta), & -\theta_1 \leqslant \theta \leqslant 0 \end{cases} \qquad (n = 1, 2).$$

Now, with known $A(\zeta)$ and $B(\zeta)$, from system (9) one can find $\Phi_1'(\zeta)$ and $\Phi_2'(\zeta)$ in the form of indefinite quadratures. The arbitrary constants will be determined from the conditions of analyticity of these functions in the elastic space.

3. In the case of the second fundamental problem of the theory of elasticity, the initial expressions are (5). Calculations analogous to those carried out above lead to the formulas

$$C(\zeta) = \frac{1}{2\pi i}\frac{1}{X_1(\zeta)}\int_L k_3(\theta)e^{-i\theta/2}X_1^+(\sigma)\frac{\sigma+\zeta}{\sigma-\zeta}\frac{d\sigma}{\sigma}+$$

$$+\frac{1}{2\pi i}\frac{1}{X_2(\zeta)}\int_L k_4(\theta)e^{-i\theta/2}X_2^+(\sigma)\frac{\zeta(\sigma+R)+\sigma(\sigma-R)}{\sigma^2(\sigma-\zeta)}\,d\sigma;$$

$$D(\zeta) = \frac{1}{2\pi i}X_1(\zeta)\int_L \frac{l_3(\theta)e^{-i\theta/2}}{X_1^+(\sigma)}\frac{\sigma+\zeta}{\sigma-\zeta}\,d\sigma + \frac{1}{2\pi i}\int_L \frac{2l_4(\theta)e^{-i\theta/2}\,d\theta}{\sigma-\zeta}.$$

Here

$$2k_3(\theta) = k_1^*(\theta)+k_2^*(\theta), \qquad 2k_4(\theta) = -k_1^*(\theta)-k_2^*(\theta),$$

where

$$k_n^*(\theta) = \begin{cases} k_n(\theta), & 0 \leqslant \theta \leqslant \theta_1, \\ k_n(-\theta), & -\theta_1 \leqslant \theta \leqslant 0 \end{cases} \qquad (n=1,2);$$

$$k_1(\theta) = \psi(2Gu^+); \qquad k_2(\theta) = \psi(2Gu^-)$$

$$2l_3(\theta) = l_1^*(\theta)-l_2^*(\theta), \qquad 2l_4(\theta) = l_1^*(\theta)+l_2^*(\theta),$$

where

$$l_n^*(\theta) = \begin{cases} l_n(\theta), & 0 \leqslant \theta \leqslant \theta_1, \\ l_n(-\theta), & -\theta_1 \leqslant \theta \leqslant 0 \end{cases} \qquad (n=1,2);$$

$$l_1(\theta) = \varphi(2Gv^+); \qquad l_2(\theta) = \varphi(2Gv^-).$$

The functions $C(\zeta)$ and $D(\zeta)$ are related to $\Phi_1(\zeta)$ and $\Phi_2(\zeta)$ by the relations

$$C(\zeta) = (R^2+\zeta^2)\zeta F''(\zeta) + (9-8\nu)\zeta^2 F'(\zeta) + (3-4\nu)\zeta F(\zeta) + \zeta\Phi_2'(\zeta);$$

$$D(\zeta) = (R^2 + \zeta^2)F'(\zeta) + \zeta F(\zeta) + \Phi_2(\zeta);$$

$$\Phi_1'(\zeta) = 2\zeta F'(\zeta) + F(\zeta).$$

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Note: Figure translations are in progress. See original paper for figures.