

# ON THE THEORY OF $\backslash(2(2s+1)\backslash)$ - COMPONENT FREE QUANTIZED SPINOR FIELDS

MATHEMATICAL PHYSICS

1968

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**Abstract**

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UDC 530.145

*MATHEMATICAL PHYSICS*

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## ON THE THEORY OF $2(2s + 1)$ -COMPONENT FREE QUANTIZED SPINOR FIELDS

*(Presented by Academician N. N. Bogolyubov, 14 XI 1966)*

In the present paper we give a group-theoretic and functional formulation of the formalism of  $2(2s + 1)$ -component free quantized spinor fields <sup>(1,2)</sup>, with the aim of subsequently studying the covariant structure of the elements of the causal  $S$ -matrix for arbitrary spin.

To define  $2(2s + 1)$ -component free spinor fields as operator-valued generalized functions, we introduce into consideration a rigged Hilbert space <sup>(3)</sup>  $D_S \subset H_{L_2} \subset D'_S$ , induced by a unitary irreducible representation  $[ms]$  of the spinor Poincaré group  $\widetilde{P}_+^\uparrow$  for a particle of mass  $m$  and spin  $s$ .

The states of the system are described by vector-valued generalized functions from the space

$$D'_S = \bigoplus_{n=0}^{\infty} D_{S(\bar{\Omega}_n^+)}^{(n)}[ms]_n, \quad (1)$$

i.e., the space of linear continuous functionals in the nuclear space <sup>(3)</sup> of Fock  $D_S$ . These vector-valued generalized functions have the form

$$|\rho(\Phi)\rangle = \sum_{n=0}^{\infty} |\rho^{(sm)n}(\Phi)^{(sm)n}\rangle = \sum_{n=0}^{\infty} \sum_{(s_3)^n} |\rho_{(s_3)^n}^{(sm)n}(\Phi)_{(s_3)^n}^{(sm)n}\rangle \quad (2)$$

for any element  $\Phi = \{\Phi_{(s_3)^n}^{(sm)n}(p)_n\}_0^\infty \in D_S$ , with components of the function  $\Phi_{(s_3)^n}^{(sm)n}(p) \in S(\bar{\Omega}_n^+)$ , where  $S(\bar{\Omega}_n^+)$  is the space of basic functions, where

$$\bar{\Omega}_n^+ = \bigotimes_{i=1}^n \bar{\Omega}_{m_i}^+, \quad \bar{\Omega}_{m_i}^+ = \{p_i \in \widetilde{R}^4; p_i^2 = (p_i^0)^2 - (\vec{p}_i)^2 = m_i^2, p_i^0 > 0, i = 1, \dots, n\};$$

$$|\rho_{(s_3)^n}^{(sm)n}(\Phi)_{(s_3)^n}^{(sm)n}\rangle$$

are generalized basis vectors of the canonical system of the continuous representation

$$[ms]_n = \prod_{i=1}^n [m_i s_i]$$

of the spinor Poincaré group  $\widetilde{P}_+^\uparrow$ ;  $s_3^i = -s_i, \dots, s_i, i = 1, \dots, n$ , are the eigenvalues of the third component of the spin-projection operator. On these generalized basis vectors is spanned the  $n$ -particle subspace  $D_{S(\widetilde{\Omega}_n^+)}^{(n)'} [ms]_n$ —a Hilbert space of generalized states. The transformation properties of the generalized vectors of the canonical system with respect to a unitary transformation  $U(a, A)$  of the Poincaré group  $\widetilde{P}_+^\uparrow$  are determined according to

$$U(a, A) |\rho_{(s_3)^n}^{(sm)n}(\Phi)_{(s_3)^n}^{(sm)n}\rangle = |\rho_{(s_3)^n}^{(sm)n}(U(a, A)\Phi)_{(s_3)^n}^{(sm)n}\rangle, \quad (3)$$

where

$$\begin{aligned} & (U(a, A)\Phi)_{(s_3)^n}^{(sm)n}(p)_n = \\ & = \exp \left[ i \sum_{j=1}^n a \cdot p_j \right] \sum_{(s_3')^n} \bigotimes_{j=1}^n D_{s_j' s_j}^{s_j m_j} [R^{-1}(A, p_j)] \Phi_{(s_3')^n}^{(sm)n}(A^{-1}p)_n. \end{aligned} \quad (4)$$

where

$$\begin{aligned} R^{-1}(A, p) &= (p \cdot \tilde{\sigma}/m)^{1/2} A^{-1} (A^{-1}p \cdot \sigma/m)^{1/2} \in SU(2, C); \quad \sigma = (\sigma_0, \vec{\sigma}); \\ \tilde{\sigma} &= (\sigma_0 - \vec{\sigma}) \end{aligned}$$

are the Pauli matrices;

$$(p \cdot \sigma/m)^{1/2} = [2m(p^0 + m)]^{-1/2} [(m + p^0) + \vec{p} \cdot \vec{\sigma}] \in SL(2, C)$$

is the Lorentz transformation to the rest frame, i.e.

$$(p \cdot \sigma/m)^{1/2} \vec{p} = p; \quad \vec{p} = (m, 0);$$

$D^{sm}[R^{-1}(A, p)]$  is the usual  $(2s+1) \times (2s+1)$ -dimensional matrix representation of the Wigner rotation <sup>(4)</sup>  $R^{-1}(A, p) \in SU(2C)$ . In formula (4) it is assumed that

$$(Ap)^\mu = \Lambda(A)^\mu{}_\nu p^\nu$$

for  $A \in SL(2, C)$  and  $\Lambda(A) \in L_+^\uparrow$ .

We note that the multiparticle generalization of the basis vector

$$|\rho_{(s_3)^n}^{(sm)n}(\Phi)_{(s_3)^n}^{(sm)n}\rangle$$

is obtained on the basis of the Schwartz kernel theorem <sup>(6)</sup> by means of the direct product of one-particle generalized basis vectors

$$|\rho_i^{s_i m_i}(\Phi)_{s_i}^{s_i m_i}\rangle$$

for arbitrary

$$\Phi_{s_i}^{s_i m_i}(p_i) \in D_{s(2m_i)}^{(1)}[m_i s_i], \quad i = 1, \dots, n,$$

which are common generalized eigenvectors of the operators of 4-momenta  $\widehat{P}^\mu$  and of the 3rd component of the projection of the spin operator  $\widehat{S}_3(p)$ , defined by the relation

$$\widehat{S}_j(p) = \frac{1}{2m} \sum_{\mu, \nu, \sigma, \lambda=0}^3 \varepsilon_{\mu\nu\sigma\lambda} (p \cdot \tilde{\sigma}/m)_j^{1/2\mu} P^\nu M^{\sigma\lambda}. \quad (5)$$

We now introduce into consideration the operator-valued generalized creation functions

$$a_{s_3}^{*sm}(\tilde{f}_{s_3})$$

and annihilation functions

$$a_{s_3}^{sm}(\tilde{f}_{s_3})$$

of a particle of mass  $m$  and spin  $s$ , and the corresponding operator-valued generalized functions

$$b_{s_3}^{*sm}(\tilde{g}_{s_3})$$

and

$$b_{s_3}^{sm}(\tilde{g}_{s_3})$$

for an antiparticle, where

$$\tilde{f}_{s_3}(p), \tilde{g}_{s_3}(p) \in S(\overline{\Omega}_m^+),$$

defined in the canonical system of generalized basis vectors, transforming according to the unitary representations

$$D^{sm}[R(A, p)] \quad \text{and} \quad D^{sm}[R^{-1}(A, p)]$$

of the spinor Poincaré group  $\widehat{P}_+^\uparrow$ . Analogously one may introduce one-particle operator-valued generalized functions

$$\hat{a}_\alpha^{*sm}(\tilde{f}^\alpha), \quad \hat{a}_\alpha^{sm}(\tilde{f}^\alpha), \quad \hat{b}_\alpha^{*sm}(\tilde{g}^\alpha), \quad \hat{b}_\alpha^{sm}(\tilde{g}^\alpha),$$

$\alpha = -s, s$ , defined in the spinor system of generalized basis vectors, transforming according to the nonunitary finite-dimensional representations

$$D^{sm}(A), \quad D^{sm}(A)^{-1+}, \quad D^{sm}(A)^{-T} \quad \text{and} \quad D^{sm}(A)^*$$

of the group  $SL(2, C)$ .

Between these two classes of operator-valued generalized functions the following relations hold. For example <sup>(5)</sup>:

$$\sum_{\alpha} \hat{a}_{\alpha}^{sm}(\tilde{f}^{\alpha}) = \sum_{s_3} a_{s_3}^{sm}(\tilde{g}_{s_3}), \quad (6)$$

where

$$\begin{aligned} \tilde{g}_{s_3}(p) &= \sum_{\alpha} D_{\alpha s_3}^{sm} [(p \cdot \sigma / m)^{1/2}] \tilde{f}^{\alpha}(p); \\ \sum_{\alpha} \hat{b}_{\alpha}^{*sm}(\tilde{f}_{-}^{\alpha}) &= \sum_{s_3} b_{s_3}^{*sm}(\tilde{h}_{s_3}), \end{aligned} \quad (7)$$

where

$$\begin{aligned} \tilde{h}_{s_3}(p) &= \sum_{\alpha} D_{\alpha s_3}^{sm} [(p \cdot \sigma / m)^{1/2} \varepsilon^{-1}] \tilde{f}^{\alpha}(-p), \\ p^0 &= \sqrt{\vec{p}^2 + m^2} \end{aligned}$$

for arbitrary

$$\tilde{f}^{\alpha}(p), \quad \tilde{f}_{-}^{\alpha}(p) = \tilde{f}^{\alpha}(-p) \in S(\overline{\Omega}_m^+).$$

In order to formulate the condition of causality or local commutativity, it is evidently necessary to define quantities with causally independent supports. To this end we introduce into consideration  $(2s + 1)$ -component spinor fields

$$\varphi_{\alpha}^{sm}(x) \quad \text{and} \quad \chi_{sm}^{\dot{\alpha}}(x),$$

respectively in the  $(s, 0)$ - and  $(0, s)$ -representations, regarded as operator-valued generalized functions, i.e. to each function

$$f^{\alpha}(x) \in S(R^4)$$

there is put in corres-

there corresponds the linear operator

$$\begin{aligned} \varphi_{\alpha}^{sm}(f^{\alpha}) &= \sum_{\alpha=-s,s} \varphi_{\alpha}^{sm}(f^{\alpha}) = 2\pi \sum_{\alpha=-s,s} [\hat{a}_{\alpha}^{sm}(\tilde{f}_{-}^{\alpha}) + \hat{b}_{\alpha}^{*sm}(\tilde{f}_{\alpha}^{\alpha})] = \\ &= \sqrt{2\pi} \sum_{s_3} [a_{s_3}^{sm}(\tilde{g}_{s_3}) + b_{s_3}^{*sm}(\tilde{h}_{s_3})] \quad \text{in } D_S, \end{aligned} \quad (8)$$

where  $(\Phi, \varphi_{\alpha}^{sm}(x)\psi) \in S'(R^4)$  for  $\Phi, \psi \in D_S$ . The field  $\chi_{sm}^{\dot{\alpha}}(x)$  has an analogous representation.

The spinor  $(2s + 1)$ -component fields  $\varphi_{\alpha}^{sm}(x)$  and  $\chi_{sm}^{\dot{\alpha}}(x)$  satisfy the wave equations

$$(-i)^{2s} \varphi_{\alpha}(\hat{\partial}_s^{\alpha\dot{\alpha}} g_{\dot{\alpha}}) = \chi^{\dot{\alpha}}(g_{\dot{\alpha}}), \quad (-i)^{2s} \chi^{\dot{\alpha}}(\hat{\partial}_{\dot{\alpha}\alpha} f^{\alpha}) = \varphi_{\alpha}(f^{\alpha}), \quad (9)$$

$$\varphi_\alpha((\square - m^2)f^\alpha) = 0, \quad \chi^\alpha((\square - m^2)g_\alpha) = 0 \quad (10)$$

for arbitrary  $f^\alpha(x), g_\alpha(x) \in S(R^4)$ , where

$$\hat{\partial}^{ss} = \begin{pmatrix} 0 & \hat{\partial}_{\alpha\dot{\alpha}}^{ss} \\ \hat{\partial}_{s\dot{s}}^{\dot{\alpha}\alpha} & 0 \end{pmatrix} = \begin{pmatrix} 0 & D_{\alpha\dot{\alpha}}^{sm}(\tilde{\sigma} \cdot \partial / m) \\ D_{\alpha\dot{\alpha}}^{sm}(\sigma \cdot \partial / m) & 0 \end{pmatrix}. \quad (11)$$

Bearing in mind that any interaction preserving parity must include both the  $\varphi_\alpha^{sm}(x)$ -field and the  $\chi_{sm}^\alpha(x)$ -field, we introduce for consideration the  $2(2s + 1)$ -component spinor fields  $\psi^{sm}(x)$  and  $\bar{\psi}^{sm}(x)$ , regarded as operator-valued generalized functions; that is, to each  $2(2s + 1)$ -component regular spinor

$$F(x) = \left\{ F_\chi(x) = \begin{pmatrix} g_\alpha(x) \\ f^\alpha(x) \end{pmatrix} \right\}, \quad \bar{F}_\chi(x) = F_\chi^*(x) \quad (12)$$

with components  $g_\alpha(x), f^\alpha(x) \in S(R^4)$  there are assigned linear operators

$$\psi^{sm}(\bar{F}) = \sum_{\chi=(\alpha,\dot{\alpha})} \psi_\chi^{sm}(\bar{F}_\chi) = \sum_{\alpha=-s,s} [\varphi_\alpha^{sm}(f^\alpha) + \chi_{sm}^{\dot{\alpha}}(g_\alpha)], \quad (13)$$

$$\bar{\psi}^{sm}(F) = \sum_{\chi=(\alpha,\dot{\alpha})} \bar{\psi}_\chi^{sm}(F_\chi) = \sum_{\alpha=-s,s} [\varphi_{\dot{\alpha}}^{sm}(f^\alpha) + \chi_{sm}^\alpha(g_\alpha)], \quad (14)$$

defined in  $D_S$ , and moreover  $(\Phi, \psi^{sm}(x)\psi) \in S'(R^4)$  and  $(\Phi, \bar{\psi}^{sm}(x)\psi) \in S'(R^4)$  for  $\Phi, \psi \in D_S$ . These fields satisfy the wave equations

$$(-i)^{2s} \psi^{sm}(\bar{F} \hat{\partial}^{ss}) = \psi^{sm}(\bar{F}), \quad \psi^{sm}((\square - m^2)\bar{F}) = 0; \quad (15)$$

$$(i)^{2s} \bar{\psi}^{sm}(\hat{\partial}^{*sm} F) = \bar{\psi}^{sm}(F), \quad \bar{\psi}^{sm}((\square - m^2)F) = 0 \quad (16)$$

and the commutation relations for free fields.

The spinor  $2(2s + 1)$ -component fields  $\psi^{sm}(x)$  and  $\bar{\psi}^{sm}(x)$  transform according to a nonunitary finite-dimensional representation of the spinor Poincaré group  $\tilde{\mathcal{P}}_+^\dagger$  as follows:

$$U(a, A)\psi^{sm}(\bar{F})U^{-1}(a, A) = \psi^{sm}(U(a, A)\bar{F}), \quad (17)$$

where

$$(U(a, A)\bar{F})(x) = \{U(a, A)\bar{F}_\chi(x)\}_{\chi=(\alpha,\dot{\alpha})}$$

$$= \left\{ \sum_{\chi'} \bar{F}_{\chi'} [A^{-1}(x-a)] D_{\chi'\chi}^{sm}(A) \right\}; \quad (18)$$

$$D^{sm}(A) = \begin{pmatrix} D^{sm}(A)^{-1} & 0 \\ 0 & D^{sm}(A)^+ \end{pmatrix} \quad (19)$$

for arbitrary  $\bar{F}(x) \in S(R^4)$  and  $A \in SL(2, C)$ .

The  $P$ -,  $T$ - and  $C$ -invariance of these fields can be expressed by means of the functional relations:

$$U_P \psi^{sm}(\bar{F}) U_P^{-1} = \xi_P \psi^{sm}(U_P \bar{F}), \quad (20)$$

where

$$(U_P \bar{F})(x) = \{U_P \bar{F}_{\chi}(x)\}_{\chi=(a,\dot{a})} = \left\{ \sum_{\chi'} \bar{F}_{\chi'}(Px) R_{\chi'\chi} \right\}, \quad (21)$$

$$R = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \bar{F}(x) \in S(R^4)$$

( $P$ -invariance; for  $(\xi_P)^2 = 1$ ,  $U_P^2 = 1$ );

$$\bar{U}_T \psi^{sm}(\bar{F}) U_T^{-1} = \xi_T \bar{\psi}^{sm}(\bar{U}_T \bar{F}), \quad (22)$$

where

$$(\bar{U}_T \bar{F})(x) = K^T R^T \bar{F}^T(Tx) \in S(R^4), \quad (23)$$

$$K^T = \begin{pmatrix} D^s(\varepsilon) & 0 \\ 0 & D^s(\varepsilon) \end{pmatrix}, \quad D_{s'_3 s_3}^s(\varepsilon) = (-)^{s+s'_3} \delta_{s'_3, -s_3}$$

( $T$ -invariance; for  $|\xi_T|^2 = 1$ ,  $U_T^2 = (-)^{2s}$ );

$$U_C \psi^{sm}(\bar{F}) U_C^{-1} = \xi_C \psi^{sm}(U_C \bar{F}), \quad (24)$$

where

$$(U_C \bar{F})(x) = \{(U_C \bar{F})_{\chi}(x)\}_{\chi=(a,\dot{a})} = \left\{ \sum_{\chi'} C_{\chi'\chi}^T \bar{F}_{\chi'}^T(x) \right\}, \quad (25)$$

$$C = \begin{pmatrix} D^s(\varepsilon) & 0 \\ 0 & (-)^{2s} D^s(\varepsilon) \end{pmatrix} \quad (26)$$

( $C$ -invariance; for  $|\xi_C|^2 = 1$ ,  $U_C^2 = 1$ ).

One can define Green' s functions for these fields.

Taking this opportunity, I express my deep gratitude to N. N. Bogolyubov, A. N. Tavkhelidze, Nguyen Van Hieu, I. Todorov, B. V. Medvedev-Vereshchagin and A. V. Efremov for useful discussions and valuable remarks.

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Received  
19 IX 1966

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*Note: Figure translations are in progress. See original paper for figures.*

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